THE REPRESENTATIONS OF CYCLOTOMIC BMW ALGEBRAS

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Abstract. In this paper, we prove that the cyclotomic BMW algebras $\mathscr{B}_{2p+1,n}$ are cellular in the sense of [16]. We also classify the irreducible $\mathscr{B}_{2p+1,n}$-modules over a field.

1. Introduction

Let r, n be two positive integers. Haering–Oldenburg [17] introduced a class of finite dimensional algebras $\mathscr{B}_{r,n}$ called cyclotomic BMW algebras in order to study the link invariants. Such algebras are associative algebras over a commutative ring R, which are the cyclotomic quotients of affine BMW algebras in [17] and [13].

Motivated by Ariki, Mathas and Rui's work on cyclotomic Nazarov–Wenzl algebras [3], we began to study $\mathscr{B}_{r,n}$ in August of 2004. By using the method given in section 3 in [3], we constructed all possible irreducible modules for $\mathscr{B}_{r,2}$.[1] A by-product is the definition of **u**-admissible conditions given in Definition 3.15 for certain parameters in the ground ring R. We remark that there are two papers [14] and [24] in Arxiv which were posted at the end of 2006. In those papers, there are some results for **u**-admissible conditions.

In this paper, we will construct a class of rational functions, which are similar to those in [3] for cyclotomic Nazarov–Wenzl algebras. Such rational functions can be used to construct the seminormal representations for $\mathscr{B}_{r,n}$ under certain conditions given in Lemma 4.6 and Assumption 4.18. By Wedderburn–Artin Theorem on the semisimple finite dimensional algebras, we prove that the rank of $\mathscr{B}_{r,n}$ is no less than $r^n(2n-1)!!$ if $\mathscr{B}_{r,n}$ is free. In order to prove that $\mathscr{B}_{r,n}$ is free over R with rank $r^n(2n-1)!!$, we have to find a subset S of $\mathscr{B}_{r,n}$ whose cardinality is $r^n(2n-1)!!$ such that $\mathscr{B}_{r,n}$ is generated by S as R-module. If so, then $\mathscr{B}_{r,n}$ is free over R with rank $r^n(2n-1)!!$ as required.

In order to construct such a subset S, we construct a class of quotient modules of $\mathscr{B}_{r,n}$. Such modules can be used to construct a filtration of two–sided ideals of $\mathscr{B}_{r,n}$. Therefore, we can lift the set of generators for all such quotient modules to get a set of generators for R-module $\mathscr{B}_{r,n}$. Together with our previous results on the seminormal representations for $\mathscr{B}_{r,n}$, we prove that $\mathscr{B}_{r,n}$ is a cellular algebra in the sense of [16]. In particular, $\mathscr{B}_{r,n}$ is free over R with rank $r^n(2n-1)!!$ as required. For some technique reasons, we have to assume that r is odd when we construct the quotient modules for $\mathscr{B}_{r,n}$. Using the results on the representation theory of cellular algebras given in [16], we find the relationship between the irreducible modules for Ariki–Koike algebras and $\mathscr{B}_{r,n}$. This will enable us to classify the irreducible $\mathscr{B}_{r,n}$–modules over a field.

We organize the paper as follows. In section 2, we recall the definition of $\mathscr{B}_{r,n}$ over a commutative ring. In section 3, we give all possible irreducible modules for $\mathscr{B}_{r,2}$. We also give the definition of **u**-admissible conditions. Under this assumption together with two conditions in Lemma 4.6 and Assumption 4.18, we construct the seminormal forms for $\mathscr{B}_{r,n}$ in section 4. In section 5, we construct the quotient modules for $\mathscr{B}_{r,n}$ for all odd positive integers r. At the end of section 5, we prove

[1] This is the main result of [26].

that such a $\mathscr{B}_{r,n}$ is cellular in the sense of [16]. Finally, we classify the irreducible $\mathscr{B}_{r,n}$–modules for odd r in section 6.

When we wrote the paper, we received Dr. Shona Yu's Ph. D thesis entitled "The cyclotomic Birman-Murakami-Wenzl Algebras" [23]. In her thesis, Yu has proved that $\mathscr{B}_{r,n}$ is cellular algebra for all r by using different method. However, we could not understand why she had assumed that ω_0, one of parameters appeared in the definition of $\mathscr{B}_{r,n}$, is invertible when she constructed a subset of $\mathscr{B}_{r,n}$ which generates $\mathscr{B}_{r,n}$ as R-module.

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2. Cyclotomic BMW algebras

Throughout the paper, we fix two positive integers r, n and a commutative ring R with multiplicative identity 1_R.

Definition 2.1. Suppose that R is a commutative ring which contains $q^{\pm 1}, u_1^{\pm 1}, \dots, u_r^{\pm 1}, \varrho^{\pm 1}, \delta^{\pm 1}$ with $\delta = q - q^{-1}$. Fix $\Omega = \{\, \omega_a \mid a \in \mathbb{Z} \,\} \subseteq R$ such that

$$\omega_0 = 1 - \delta^{-1}(\varrho - \varrho^{-1}).$$

The cyclotomic BMW algebras $\mathscr{B}_{r,n}$ is the unital associative R–algebra generated by $\{\, T_i, E_i, X_j^{\pm 1} \mid 1 \le i < n \text{ and } 1 \le j \le n \,\}$ subject to the following relations:

a) $X_i X_i^{-1} = X_i^{-1} X_i = 1$ for $1 \le i \le n$.

b) (Kauffman skein relation) $1 = T_i^2 - \delta T_i + \delta \varrho E_i$, for $1 \le i < n$.

c) (braid relations)
  - *(i)* $T_i T_j = T_j T_i$ if $|i - j| > 1$,
  - *(ii)* $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, for $1 \le i < n - 1$,
  - *(iii)* $T_i X_j = X_j T_i$ if $j \ne i, i + 1$.

d) (Idempotent relations) $E_i^2 = \omega_0 E_i$, for $1 \le i < n$.

e) (Commutation relations) $X_i X_j = X_j X_i$, for $1 \le i, j \le n$.

f) (Skein relations)
  - *(i)* $T_i X_i - X_{i+1} T_i = \delta X_{i+1}(E_i - 1)$, for $1 \le i < n$,
  - *(ii)* $X_i T_i - T_i X_{i+1} = \delta (E_i - 1) X_{i+1}$, for $1 \le i < n$.

g) (Unwrapping relations) $E_1 X_1^a E_1 = \omega_a E_1$, for $a \in \mathbb{Z}$.

h) (Tangle relations)
  - *(i)* $E_i T_i = \varrho E_i = T_i E_i$, for $1 \le i \le n - 1$,
  - *(ii)* $E_{i+1} E_i = E_{i+1} T_i T_{i+1} = T_i T_{i+1} E_i$,for $1 \le i \le n - 2$.

i) (Untwisting relations)
  - *(i)* $E_{i+1} E_i E_{i+1} = E_{i+1}$ for $1 \le i \le n - 2$,
  - *(ii)* $E_i E_{i+1} E_i = E_i$, for $1 \le i \le n - 2$.

j) (Anti–symmetry relations) $E_i X_i X_{i+1} = E_i = X_i X_{i+1} E_i$, for $1 \le i < n$.

k) (Cyclotomic relation) $(X_1 - u_1)(X_1 - u_2) \cdots (X_1 - u_r) = 0$

By Definition 2.1(b)(f)(h), we have $X_i = T_{i-1} X_{i-1} T_{i-1}$ for $2 \le i \le n$. Therefore, $\mathscr{B}_{r,n}$ can also be generated by E_i, T_i, X_1 for $1 \le i \le n - 1$. We will not need this fact. What we will need is $X_i = T_{i-1} X_{i-1} T_{i-1}$ later on.

$\mathscr{B}_{r,n}$ was introduced by Haering–Oldenburg in order to study the link invariants. It was re-defined by Goodman and Hauschild as the quotient algebra of affine BMW algebra in [13]. Further, Goodman and Hauschild [13] have proved that affine BMW algebras are free over R by showing that they are isomorphic to the affine Kauffman tangle algebras. We do not recall such result since we do not need it when we discuss the $\mathscr{B}_{r,n}$ later on.

The main purpose of this paper is to develop the representation theory of $\mathscr{B}_{r,n}$ by using the method in [3].

Lemma 2.2. *There is a unique R-linear anti-isomorphism $*: \mathscr{B}_{r,n} \longrightarrow \mathscr{B}_{r,n}$ such that $T_i^* = T_i, E_i^* = E_i$ and $X_j^* = X_j$ for all positive integers $i < n$ and $j \leq n$.*

Proof. By checking the defining relations for $\mathscr{B}_{r,n}$, $*$ is an R-linear anti-automorphism if $E_iT_{i+1}T_i = E_iE_{i+1} = T_{i+1}T_iE_{i+1}$ for $1 \leq i \leq n-2$. In fact, By Definition 2.1(h)(ii), $E_{i+1}E_i = E_{i+1}T_iT_{i+1}$. So, $E_i = E_iE_{i+1}T_iT_{i+1}$ and $E_iT_{i+1}^{-1} = E_iE_{i+1}T_i$. By Definition 2.1(b) and (h)(i), $E_i(T_{i+1} - \delta + \delta E_{i+1}) = E_iE_{i+1}(T_i^{-1} + \delta - \delta E_i)$. By Definition 2.1(i)(ii) again, $E_iT_{i+1} = E_iE_{i+1}T_i^{-1}$. In other words, $E_iT_{i+1}T_i = E_iE_{i+1}$. Similarly, we can prove $E_iE_{i+1} = T_{i+1}T_iE_{i+1}$. □

Lemma 2.3. *Given positive integers $k \leq n-1$ and a.*

(1) $T_kX_k^a = X_{k+1}^aT_k + \sum_{i=1}^a \delta X_{k+1}^i(E_k - 1)X_k^{a-i}$,
(2) $T_k^{-1}X_k^a = X_{k+1}^aT_k^{-1} + \sum_{i=1}^a \delta X_{k+1}^{a-i}(E_k - 1)X_k^i$,
(3) $E_kX_k^aT_k = \varrho E_kX_k^{-a} + \delta \sum_{i=1}^a E_kX_k^{a-i}E_kX_k^{-i} - \delta \sum_{i=1}^a E_kX_k^{a-2i}$.
(4) $T_kX_k^{-a} = X_{k+1}^{-a}T_k - \sum_{i=1}^a \delta X_{k+1}^{-a+i}(E_k - 1)X_k^{-i}$,
(5) $T_k^{-1}X_k^{-a} = X_{k+1}^{-a}T_k^{-1} - \sum_{i=1}^a \delta X_{k+1}^{-i}(E_k - 1)X_k^{-a+i}$,
(6) $E_kX_k^{-a}T_k = \varrho E_kX_k^a - \delta \sum_{i=1}^a E_kX_k^{-i}E_kX_k^{a-i} + \delta \sum_{i=1}^a E_kX_k^{a-2i}$.

Proof. (1) can be proved by induction on a. (2) follows from (1) and 2.1(b). Applying anti-automorphism $*$ on (1), we get the formula for $X_k^aT_k$. Multiplying E_k on such a formula, we get (3). (4) follows from (1). (5) follows from (4) and 2.1(b). (6) follows from (4). □

Acting E_1 on both sides of $X_1^b \prod_{i=1}^r (X_1 - u_i) = 0$ for $b \in \mathbb{Z}$, and Lemma 2.3(1) for $k = 1$, respectively, we have

$$\left\{ \begin{array}{l} \sum_{s=0}^r (-1)^{r-s}\sigma_{r-s}(\mathbf{u})\omega_{s+b}E_1 = 0 \\ \omega_aE_1 = \omega_{-a}E_1 + \sum_{i=1}^a \varrho^{-1}\delta(\omega_{a-i}\omega_{-i} - \omega_{a-2i})E_1, \end{array} \right. \tag{2.4}$$

where $\sigma_i(\mathbf{u})$ is the i-th basic symmetric polynomial in $u_1, u_2, \cdots, u_r$.

If we assume that E_1 is not a torsion element, then

$$\left\{ \begin{array}{l} \sum_{s=0}^r (-1)^{r-s}\sigma_{r-s}(\mathbf{u})\omega_{s+b} = 0, \forall\, b \in \mathbb{Z}, \\ \omega_a = \omega_{-a} + \sum_{i=1}^a \varrho^{-1}\delta(\omega_{a-i}\omega_{-i} - \omega_{a-2i}). \end{array} \right. \tag{2.5}$$

Definition 2.6. The parameters $\omega_a \in R, a \in \mathbb{Z}$ are called admissible if they satisfy (2.5).

In [1], Ariki and Koike introduced $\mathscr{H}_{r,n}(\mathbf{u}) := \mathscr{H}_{r,n}$, the cyclotomic Hecke algebra of type $G(r, 1, n)$ or the Ariki-Koike algebra. By definition, it is the unital associative R-algebra generated by $y_1, \dots, y_n$ and $g_1, g_2, \dots, g_{n-1}$ subject to the following relations:

a) $(g_i - q)(g_i + q^{-1}) = 0$, if $1 \leq i \leq n-1$,
b) $g_ig_j = g_jg_i$, if $|i - j| > 1$,
c) $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$, for $1 \leq i < n-1$,
d) $g_iy_j = y_jg_i$, if $j \neq i, i+1$,
e) $y_iy_j = y_jy_i$, for $1 \leq i, j \leq n$,
f) $y_{i+1} = g_iy_ig_i$, for $1 \leq i \leq n-1$,
g) $(y_1 - u_1)(y_1 - u_2) \dots (y_1 - u_r) = 0$.

In this paper, since we are assuming that $u_1, \dots, u_r$ are invertible in R, y_i's are invertible in $\mathscr{H}_{r,n}$.

Let $\langle E_1\rangle$ be the two-sided ideal of $\mathscr{B}_{r,n}$ generated by E_1. It is not difficult to see that there is an epimorphism

$$\eta : \mathscr{H}_{r,n}(\mathbf{u}) \longrightarrow \mathscr{B}_{r,n}/\langle E_1\rangle \tag{2.7}$$

determined by: $\eta(g_i) = T_i + \langle E_1\rangle$ and $\eta(y_j) = X_j + \langle E_1\rangle$ for positive integers $i < n$ and $j \le n$.

So, any $\mathscr{B}_{r,n}$-module, which is annihilated by E_1, is an $\mathscr{H}_{r,n}(\mathbf{u})$-module. We will use this fact frequently in the next section.

3. u-admissible conditions

In this section, unless otherwise stated, we always assume that $R = \mathbb{Q}(u_1, u_2, \cdots, u_r, q)$, where $q, u_1, u_2, \cdots, u_r$ are algebraically independent over $\mathbb{Q}$. Let $\varrho^{\pm 1} \in R$, $\delta = q - q^{-1}$ and $\Omega = \{\, \omega_a \mid a \in \mathbb{Z}\,\} \subseteq R$ such that

$$\omega_0 = 1 - \delta^{-1}(\varrho - \varrho^{-1}).$$

The main purpose of this section is to construct all possible irreducible representations of $\mathscr{B}_{r,2}$ over R. We find a set of conditions on the parameters ϱ and $\{\omega_a \mid a \in \mathbb{Z}\}$, called **u**-admissible conditions, such that the dimension of the corresponding $\mathscr{B}_{r,2}$ is $3r^2$. These conditions, which are similar to those for the cyclotomic Nazarov-Wenzl algebras in [3], are exactly what we need for general n.[2]

Proposition 3.1. *Suppose that M is an irreducible $\mathscr{B}_{r,2}$–module such that $E_1 M = 0$. Then one of the following results holds.*

a) *$M = Rm$ is one dimensional and the action of $\mathscr{B}_{r,2}$ is determined by*

$$T_1 m = \varepsilon m, \quad E_1 m = 0, \quad X_1 m = u_i m, \quad \text{and} \quad X_2 m = \varepsilon^2 u_i m,$$

where $\varepsilon \in \{q, -q^{-1}\}$ and $1 \le i \le r$. In particular, up to isomorphism, there are exact $2r$ such representations.

b) *M is two dimensional and the action of $\mathscr{B}_{r,2}$ is given by*

$$T_1 \mapsto \frac{u_j}{u_j - u_i}\begin{pmatrix} \delta & q - u_i q^{-1} u_j^{-1} \\ q^{-1} - q u_i u_j^{-1} & -\delta u_i u_j^{-1}\end{pmatrix}$$
$$E_1 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$$
$$X_1 \mapsto \begin{pmatrix} u_i & 0 \\ 0 & u_j\end{pmatrix}$$
$$X_2 \mapsto \begin{pmatrix} u_j & 0 \\ 0 & u_i\end{pmatrix}$$

where $1 \le i, j \le r$ with $i \ne j$. In particular, up to isomorphism there are exact $\binom{r}{2}$ such representations.

Proof. It follows from (2.7) that any irreducible $\mathscr{B}_{r,2}$-module M has to be an irreducible $\mathscr{H}_{r,2}$-module if $E_1 M = 0$. By the results for $\mathscr{H}_{r,2}$ in [1], M has to be one of the modules given in (a) and (b). By direct computation, both (a) and (b) do define the $\mathscr{B}_{r,2}$-modules with trivial action of E_1 on them. $\square$

Proposition 3.2. *Suppose $\omega_0 \ne 0$. There is a unique irreducible $\mathscr{B}_{r,2}$–module M such that $E_1 M \ne 0$. Moreover, $\dim_R M \le r$. If $d = \dim_R M$, then there exists a basis $\{m_1, m_2, \ldots, m_d\}$ of M and scalars $\{v_1, \ldots, v_d\} \subseteq \{u_1, \ldots, u_r\}$, with $v_i \ne v_j$ whenever $i \ne j$, such that for $1 \le i \le d$ the following hold:*

a) *$X_1 m_i = v_i m_i$ and $X_2 m_i = v_i^{-1} m_i$,*

b) *$E_1 m_i = \gamma_i (m_1 + \cdots + m_d)$*

[2] The results given in this section are the main results of [26]. Similar results can be found in [24]. We remark the method we are using here is almost the same as that used in [3].

c) $T_1 m_i = \dfrac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{j\neq i} \dfrac{\delta\gamma_i}{v_i v_j - 1} m_j,$

where $\omega_a = \sum_{j=1}^{d} v_j^a \gamma_j$, $a \in \mathbb{Z}$, *and*

(1) $\gamma_i = (\gamma_d(v_i) + \delta^{-1}\varrho(v_i^2 - 1)\prod_{j\neq i} v_j)\prod_{j\neq i} \frac{v_i v_j - 1}{v_i - v_j}$, *where* $\gamma_d(z) = 1$ *for* $2 \nmid d$ *and* $-z$, *otherwise.*

(2) $\varrho^{-1} = \alpha\prod_{l=1}^{d} v_l$ *where* $\alpha \in \{1, -1\}$ *if* $2 \nmid d$ *and* $\alpha \in \{q^{-1}, -q\}$, *otherwise.*

(3) $\omega_0 = \delta^{-1}\varrho(\prod_{l=1}^{d} v_l^2 - 1) + 1 - \frac{(-1)^d + 1}{2}\prod_{i=1}^{d} v_i$.

Conversely, if γ_j, ϱ^{-1}, $\omega_a = \sum_{j=1}^{d} v_j^a \gamma_j$, $a \in \mathbb{Z}$, *are defined as above, then (a)-(c) define an irreducible* $\mathscr{B}_{r,2}$*-module with* $E_1 M \neq 0$.

Proof. The result can be proved by arguments similar to those for the cyclotomic Nazarov-Wenzl algebras in [3]. We include a proof here.

By direct computation, one can verify that the R-module N generated by $\{X_1^\alpha X_2^\beta, X_1^\alpha X_2^\beta T_1, X_1^\alpha E_1 X_1^\beta | 0 \le \alpha, \beta \le r - 1\}$ is a right $\mathscr{B}_{r,2}$-module. Since $1 \in N$, $N = \mathscr{B}_{r,2}$. In particular, $\mathscr{B}_{r,2}$ is of finite dimension. So is any irreducible $\mathscr{B}_{r,2}$-module.

Suppose that M is an irreducible $\mathscr{B}_{r,2}$–module such that $E_1 M \neq 0$ and $d = \dim_R M$. We first show that (a)–(c) hold.

Since $u_1, \dots, u_r$ are pairwise distinct, we can fix a basis $\{m_1, \dots, m_d\}$ of M consisting of eigenvectors for X_1. Write $X_1 m_i = v_i m_i$, for some $v_i \in \{u_1, \cdots, u_r\}$. We remark that we will prove that $v_i \neq v_j$ whenever $i \neq j$.

Since we are assuming that $\omega_0 \neq 0$, we define $f = \frac{1}{\omega_0} E_1$. Then $f^2 = f$, $E_1 f = f E_1 = \omega_0 f$, and $f \neq 0$. Also, we have $fM \neq 0$ since $E_1 M \neq 0$.

Fix an element $0 \neq m \in fM$. Then $E_1 m = \omega_0 m$ and $T_1 m = \varrho m$ (since $T_1 E_1 = \varrho E_1$). As $X_1 X_2 E_1 = E_1$, $X_1 X_2 E_1 m = \omega_0 m$. Therefore, $X_1 X_2 m = m$.

We consider the R-module M' generated by $\{m, X_1 m, \dots, X_1^{d-1} m\}$. Since $E_1 X_1^k m = E_1 X_1^k f m = \frac{1}{\omega_0} E_1 X_1^k E_1 m = \frac{\omega_k}{\omega_0} E_1 m = \omega_k m$, M' is stable under the action of E_1. By Lemma 2.3(3) for $k = 1$ and Lemma 2.2,

$$\begin{aligned} T_1 X_1^a m &= \frac{1}{\omega_0} T_1 X_1^a E_1 m \\ &= \frac{1}{\omega_0}(\varrho X_1^{-a} + \sum_{i=1}^{a} \delta(\omega_{a-i} X_1^{-i} - X_1^{a-2i})) E_1 m \\ &= \varrho X_1^{-a} m + \sum_{i=1}^{a} \delta(\omega_{a-i} X_1^{-i} m - X_1^{a-2i} m) \end{aligned}$$

Note that $(X_1 - v_1)(X_1 - v_2)\cdots(X_1 - v_d) = 0$ on M. Each X_1^a for $a \in \mathbb{Z}$ can be written as a linear combination of X_1^b with $0 \le b \le d - 1$. Therefore, M' is closed under the action of T_1. Note that $X_2 = T_1 X_1 T_1$. So, M' is closed under the action of X_2. We have $M' = M$ since M is irreducible. In particular, by $\dim M = d$, $\{m, X_1 m, \cdots, X_1^{d-1} m\}$ is a basis of M. Since $E_1 X_1^k m = \omega_k m$, $E_1 M = Rm$.

Write $m = \sum_{i=1}^{d} r_i m_i$, for some $r_i \in R$. Suppose that $r_i = 0$ for some i. Then

$$\prod_{\substack{1 \le j \le d \\ j \neq i}} (X_1 - v_j) \cdot m = 0.$$

This contradicts the linear independence of $\{m, X_1 m, \dots, X_1^{d-1} m\}$; hence, $r_i \neq 0$ for $i = 1, \dots, d$. By rescaling the m_i if necessary, we do assume that

$$m = m_1 + m_2 + \cdots + m_d \tag{3.3}$$

in the following.

By the previous arguments, all of the eigenvalues $\{v_1, \dots, v_d\}$ of X_1 must be distinct. It is easy to verify that $X_1 X_2$ is central in $\mathscr{B}_{r,2}$, $X_1(X_1X_2)m_i = (X_1X_2)X_1 m_i$. So X_1X_2 acts on m_i as a scalar since u_i's are algebraic independent. On the other hand, since $X_1X_2 m = m$, by (3.3), X_1X_2 acts as 1 on M. Therefore, $X_2 m_i = X_1^{-1} m_i = v_i^{-1} m_i$, for $i = 1, \dots, d$, proving (a).

Since $u_1, \dots, u_r$ are algebraically independent, $v_i^2 - 1$ and $v_i v_j - 1$, for $i \neq j$, are invertible in R. So the formula in part (c) makes sense.

As $E_1 M = Rm$, we can define elements $\gamma_i \in R$ by

$$E_1 m_i = \gamma_i m = \gamma_i(m_1 + \cdots + m_d), \qquad \text{for } i = 1, \dots, d.$$

Write $T_1 m_i = \sum_{j=1}^d c_{ji} m_j$. By Definition 2.1(f) for $i = 1$, we have

$$T_1 X_2 m_i - X_1 T_1 m_i = \delta X_2 m_i - \delta E_1 X_2 m_i.$$

So,

$$\sum_{j=1}^d v_i^{-1} c_{ji} m_j - \sum_{j=1}^d c_{ji} v_j m_j = \delta v_i^{-1} m_i - \delta v_i^{-1} \gamma_i m.$$

Comparing the coefficients of m_j on both sides of the above equation, we have

$$c_{ji} = \frac{\delta(\gamma_i - \delta_{ij})}{v_i v_j - 1}, \tag{3.4}$$

where δ_{ij} is the Kronecker function. Therefore,

$$T_1 m_i = \frac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{\substack{1 \leq j \leq d \\ j \neq i}} \frac{\delta \gamma_i}{v_i v_j - 1} m_j.$$

This proves (c).

Since $\varrho E_1 = T_1 E_1$, we have, for any positive integer $i \leq d$:

$$\begin{aligned}
\varrho \gamma_i \sum_{j=1}^d m_j = \varrho E_1 m_i = T_1 E_1 m_i &= T_1(\gamma_i \sum_{j=1}^d m_j) \\
&= \gamma_i \sum_{j=1}^d \Big(\frac{\delta(\gamma_j - 1)}{v_j^2 - 1} m_j + \sum_{\substack{1 \leq k \leq d \\ k \neq j}} \frac{\delta \gamma_j}{v_j v_k - 1} m_k \Big) \\
&= \gamma_i \Big(\sum_{j=1}^d \frac{\delta(\gamma_j - 1)}{v_j^2 - 1} m_j + \sum_{j=1}^d \sum_{\substack{1 \leq k \leq d \\ k \neq j}} \frac{\delta \gamma_k}{v_j v_k - 1} m_j \Big)
\end{aligned}$$

Since $E_1 M \neq 0$, there is at least a non-zero γ_i. Therefore,

$$\varrho \sum_{j=1}^d m_j = \sum_{j=1}^d \frac{\delta(\gamma_j - 1)}{v_j^2 - 1} m_j + \sum_{j=1}^d \sum_{\substack{1 \leq k \leq d \\ k \neq j}} \frac{\delta \gamma_k}{v_j v_k - 1} m_j.$$

Comparing the coefficients of m_j on both sides of the equality, we have

$$\sum_{k=1}^d \frac{\gamma_k}{v_j v_k - 1} = \delta^{-1} \varrho + \frac{1}{v_j^2 - 1}, j = 1, 2, \cdots, d. \tag{3.5}$$

The system of linear equations (3.5) on γ_k has a unique solution since $\det A_d \neq 0$ where A_d is the coefficient matrix such that the (i,j)-th entry of A_d is $(v_i v_j - 1)^{-1}$. In fact, we have

$$\det A_d = \prod_{1\le k<j\le d} (v_k - v_j)^2 \prod_{1\le k,j\le d} (v_k v_j - 1)^{-1}. \tag{3.6}$$

In order to verify (3.6), we first observe that $\prod_{1\le k,j\le d}(v_k v_j - 1)\cdot \det A_d$ is a symmetric polynomial in $v_1, \dots, v_d$. So, it is divided by $v_k - v_j, k \neq j$. Therefore, there is a $f_d(v_1, \cdots, v_d) \in \mathbb{Q}[v_1, \dots, v_d]$ such that

$$\prod_{1\le k,j\le d} (v_k v_j - 1)\cdot \det A_d = f_d(v_1, \cdots, v_d) \prod_{1\le k<j\le d} (v_k - v_j)^2. \tag{3.7}$$

Comparing the coefficients of v_1 on both sides of (3.7), we know that the highest degree of v_1 in $f_d(v_1, v_2, \cdots, v_d)$ is zero. Since $f_d(v_1, v_2, \cdots, v_d)$ is a symmetric polynomial in $v_1, v_2, \cdots, v_d$, $f_d(v_1, v_2, \cdots, v_d) = c_d \in \mathbb{Q}$.

In order to determine c_d, we set $v_d = 0$ and get

$$\det A_d|_{v_d=0} = -\det A_{d-1} \prod_{i=1}^{d-1} v_i^2$$

Using (3.7) to rewrite the above equation, we have $c_d = c_{d-1}$. By induction assumption, $c_d = c_j$, $1 \le j \le d-1$. An easy computation shows that $c_1 = 1$. Thus $c_d = 1$, proving (3.6).

We have proved that the system of linear equations given in (3.5) has a unique solution. We define

$$f(z) = \frac{\gamma_d(z)}{(z^2-1)(v_j z - 1)} \prod_{l=1}^{d} \frac{v_l z - 1}{z - v_l} + \frac{\delta^{-1}\varrho}{z(v_j z - 1)} \prod_{l=1}^{d} \frac{v_l(v_l z - 1)}{z - v_l},$$

where $\gamma_d(z)$ is defined in (1) of Proposition 3.2. By residue theorem for complete non-singular curves for $f(z)$, we have

$$\sum_{k=1}^{d} \frac{1}{v_j v_k - 1}(\gamma_d(v_k) + \delta^{-1}\varrho(v_k^2 - 1)\prod_{l\neq k} v_l) \prod_{l\neq k} \frac{v_k v_l - 1}{v_k - v_l} = \delta^{-1}\varrho + \frac{1}{v_j^2 - 1}.$$

The above equalities shows that γ_k's given in (1) in Proposition 3.2 satisfy (3.5). We remark that the left (resp. right) side of the above equality can be interpreted as $\sum_{k=1}^{d} Res_{z=v_k} f(z)dz$ (resp. $-\sum\limits_{v\in I} Res_{z=v} f(z)dz$ and $I = \{\infty, \pm 1, 0\}$). This completes the proof of (b).

Now, we prove the formula about ω_a, $a \in \mathbb{Z}$. Since $E_1 m = \omega_0 m$ and $m = \sum_{i=1}^{d} m_i$, $\omega_0 = \sum_{i=1}^{d} \gamma_i$. Similarly, we have that $\omega_a = \sum_{j=1}^{d} v_j^a \gamma_j$. In order to compute ω_0, we define

$$g(z) = \frac{\gamma_d(z)}{z^2 - 1} \prod_{l=1}^{d} \frac{v_l z - 1}{z - v_l} + \frac{\delta^{-1}\varrho}{z} \prod_{l=1}^{d} \frac{v_l(v_l z - 1)}{z - v_l}.$$

Then $\omega_0 = \sum_{i=1}^{d} \gamma_i = \sum_{i=1}^{d} Res_{z=v_i} g(z)dz$. By residue theorem for complete non-singular curves for $g(z)$, $\omega_0 = -\sum_{v\in I} Res_{z=v} f(z)dz$ and $I = \{\infty, \pm 1, 0\}$. By direct computation,

$$\omega_0 = \delta^{-1}\varrho(\prod_{l=1}^{d} v_l^2 - 1) + 1 - \frac{(-1)^d + 1}{2} \prod_{l=1}^{d} v_l.$$

By solving the equation $(\omega_0 - 1)\delta = \varrho^{-1} - \varrho$, we get ρ as required.

We next show that M is uniquely determined, up to isomorphism. Suppose that $\mathscr{B}_{r,2}$ has another irreducible module of dimension d' upon which E_1 acts non–trivially. Then, by the previous arguments,

$$\omega_0 = \delta^{-1}\varrho'(\prod_{l=1}^{d'} w_l^2 - 1) + 1 - \frac{(-1)^{d'}+1}{2}\prod_{l=1}^{d'} w_l.$$

for some $\{ w_1, \ldots, w_{d'} \} \subseteq \{u_1, \ldots, u_r\}$. Since we are assuming that $u_1, \ldots, u_r, q$ are algebraically independent, $d' = d$, $\varrho = \varrho'$ and $w_i = v_{(i)\sigma}$, for some $\sigma \in \mathfrak{S}_d$ and $1 \le i \le d$. By (a)–(c), $M \cong M'$ as required.

Finally, it remains to verify that (a)–(c) do define a representation of $\mathscr{B}_{r,2}(\mathbf{u})$ whenever $\omega_a = \sum_{i=1}^d v_i^a \gamma_i$, for $a \in \mathbb{Z}$ and γ_i, ϱ as above. In other word, we need verify the defining relations for $\mathscr{B}_{r,2}$. More explicitly, we need verify (a), (b), (d)–(h) and (j)-(k) in Definition 2.1.

In fact, it is easy to verify (a),(e),(j),(k). Note that (d) and (g) can be verified easily by using the formula $\omega_a = \sum_{j=1}^d v_j^d \gamma_j$. (f) follows from (3.4) and (h) follows from

$$\sum_{k=1}^d \frac{\gamma_k}{v_j v_k - 1} = \delta^{-1}\varrho + \frac{1}{v_j^2 - 1}, j = 1, 2, \cdots, d.$$

Finally, we verify (b) in Definition 2.1.

We have already proved that Definition 2.1(f) holds on M. Thus, the following equalities hold in $\mathrm{End}_R(M)$:

$$\begin{cases} T_1^2 X_2 - T_1 X_1 T_1 = \delta T_1 X_2 - \varrho\delta E_1 X_2 \\ X_2 T_1^2 - T_1 X_1 T_1 = \delta X_2 T_1 - \varrho\delta X_2 E_1 \end{cases}$$

Therefore, $(T_1^2 - \delta T_1 + \varrho\delta E_1)X_2 = X_2(T_1^2 - \delta T_1 + \varrho\delta E_1)$. Since $v_i^{-1} \neq v_j^{-1}$ whenever $i \neq j$, and $X_2 m_i = v_i^{-1} m_i$, $T_1^2 - \delta T_1 + \varrho\delta E_1$ acts on $\{m_1, \cdots, m_d\}$ diagonally. So, for each positive integer $i \le d$, there is a $c_i \in R$ such that

$$(T_1^2 - \delta T_1 + \varrho\delta E_1)m_i = c_i m_i.$$

So,

$$\begin{aligned}
&(T_1^2 - \delta T_1 + \varrho\delta E_1)m_i \\
=&T_1\Big(\frac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{\substack{1\le j\le d\\ j\neq i}} \frac{\delta\gamma_i}{v_i v_j - 1} m_j\Big) \\
&- \delta\Big(\frac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{\substack{1\le j\le d\\ j\neq i}} \frac{\delta\gamma_i}{v_i v_j - 1} m_j\Big) + \varrho\delta\gamma_i \sum_{j=1}^d m_j \\
=&\frac{\delta(\gamma_i - 1)}{v_i^2 - 1}\Big(\frac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{\substack{1\le j\le d\\ j\neq i}} \frac{\delta\gamma_i}{v_i v_j - 1} m_j\Big) + \sum_{\substack{1\le j\le d\\ j\neq i}} \frac{\delta\gamma_i}{v_i v_j - 1}\Big(\frac{\delta(\gamma_j - 1)}{v_j^2 - 1} m_j \\
&+ \sum_{\substack{1\le k\le d\\ k\neq j}} \frac{\delta\gamma_j}{v_j v_k - 1} m_k\Big) - \delta\Big(\frac{\delta(\gamma_i - 1)}{v_i^2 - 1} m_i + \sum_{\substack{1\le j\le d\\ j\neq i}} \frac{\delta\gamma_i}{v_i v_j - 1} m_j\Big) + \varrho\delta\gamma_i \sum_{j=1}^d m_j \\
=&c_i m_i.
\end{aligned}$$

Comparing the coefficient of m_i on both sides of the above equality, we have

$$c_i = \delta^2 \gamma_i \sum_{j=1}^d \frac{\gamma_j}{(v_i v_j - 1)^2} + \delta^2 \frac{v_i^2 - (1 + v_i^2)\gamma_i}{(v_i^2 - 1)^2} + \varrho\delta\gamma_i.$$

Suppose

$$h(z)=\frac{\gamma_d(z)}{(z^2-1)(v_iz-1)^2}\prod_{l=1}^{d}\frac{v_lz-1}{z-v_l}+\frac{\delta^{-1}\varrho}{z(v_iz-1)^2}\prod_{l=1}^{d}\frac{v_l(v_lz-1)}{z-v_l}.$$

We use the residue theorem for complete non-singular curves for $h(z)$ and the equalities for γ_i, $1\le i\le d$, in Proposition 3.2 to compute c_i. We discuss the case $2\nmid d$ and leave the other to the reader.

By direct computation, we have

$$Res_{z=v}h(z)=\begin{cases}-\frac{1}{2}(v_i-1)^{-2}, & \text{if } v=1,\\ -\frac{1}{2}(v_i+1)^{-2}, & \text{if } v=-1,\\ \delta^{-1}\varrho, & \text{if } v=0,\end{cases}$$

and

$$\begin{cases} Res_{z=v_i^{-1}}h(z)=\frac{v_i^2}{(1-v_i^2)^2}\prod_{l\neq i}\frac{v_i-v_l}{v_iv_l-1}+\delta^{-1}\varrho\frac{v_i^2}{1-v_i^2}\prod_{l\neq i}v_l\prod_{l\neq i}\frac{v_i-v_l}{v_iv_l-1},\\ Res_{z=v_k}h(z)=\frac{1}{(1-v_iv_k)^2}\prod_{l\neq k}\frac{v_kv_l-1}{v_k-v_l}+\delta^{-1}\varrho\frac{v_k^2-1}{(1-v_iv_k)^2}\prod_{l\neq k}v_l\prod_{l\neq k}\frac{v_kv_l-1}{v_k-v_l}.\end{cases}$$

Using (1)–(2) of Proposition 3.2 to rewrite the above equalities yields

$$\begin{cases} Res_{z=v_i^{-1}}h(z) &= \frac{v_i^2}{\gamma_i(1-v_i^2)^2}-\frac{1}{\gamma_i\delta^2}\\ Res_{z=v_k}h(z) &= \frac{\gamma_k}{(1-v_iv_k)^2}.\end{cases}$$

By the residue theorem for complete non-singular curves for $h(z)$, we have

$$Res_{z\in I}h(z)+\sum_{k=1}^{d}Res_{z=v_k}h(z)=0$$

where $I=\{\pm1,0,v_i^{-1}\}$. Rewriting the above equality yields

$$\delta^{-1}\varrho=\frac{v_i^2+1}{(v_i^2-1)^2}-\frac{v_i^2}{\gamma_i(1-v_i^2)^2}\Big(1-\frac{\delta^{-2}(v_i^2-1)^2}{v_i^2}\Big)-\sum_{k=1}^{d}\frac{\gamma_k}{(1-v_iv_k)^2}.$$

So,

$$c_i=\delta\varrho\gamma_i-\frac{v_i^2+1}{(v_i^2-1)^2}\delta^2\gamma_i+\frac{v_i^2}{(1-v_i^2)^2}\delta^2+\delta^2\gamma_i\sum_{k=1}^{d}\frac{\gamma_k}{(1-v_iv_k)^2}=1.$$

This shows that Definition 2.1(b) holds on M. □

Theorem 3.8. *Suppose $R=\mathbb{Q}(u_1,u_2,\cdots,u_r,q)$ where $u_1,\dots,u_r,q$ are algebraically independent over $\mathbb{Q}$. If ϱ and $\omega_a, a\in\mathbb{Z}$, are given as in Proposition 3.2 for $d=r$, then $\mathscr{B}_{r,2}$ is semisimple over R. Moreover, $S=\{X_1^\alpha X_2^\beta, X_1^\alpha X_2^\beta T_1, X_1^\alpha E_1X_1^\beta \mid 0\le\alpha,\beta\le r-1\}$ is an R–basis of $\mathscr{B}_{r,2}$.*

Proof. By direct computation, one can verify that the R-submodule M of $\mathscr{B}_{r,2}$ generated by S is a right $\mathscr{B}_{r,2}$-module. Since $1\in S$, $M=\mathscr{B}_{r,2}$. In particular, $\dim\mathscr{B}_{r,2}\le 3r^2$.

Under the assumption, we have, by Proposition 3.2, that there is a unique irreducible $\mathscr{B}_{r,2}$-module M with $E_1M\neq 0$. By Wedderburn-Artin Theorem for semisimple finite dimensional algebras together with Proposition 3.1, we have

$$3r^2=2r+4\frac{r(r-1)}{2}+r^2=dim\mathscr{B}_{r,2}/Rad\mathscr{B}_{r,2}\le dim\mathscr{B}_{r,2}\le 3r^2,$$

where $\mathrm{Rad}\,\mathscr{B}_{r,2}$ is the Jacobson radical of $\mathscr{B}_{r,2}$. Thus all inequalities given above are equalities. In particular, $\mathscr{B}_{r,2}$ is semisimple and S is an R–basis. □

Definition 3.9. Suppose $\mathbf{x}=(x_1,x_2,\cdots,x_r)$ are variables. For any non-negative integer a, Define $Q_a(\mathbf{x}), Q'_a(\mathbf{x})$ such that

$$\begin{cases} \prod_{i=1}^r \frac{y-x_i}{x_iy-1} = \sum_{a=0}^\infty Q_a(\mathbf{x})y^a, \\ \prod_{i=1}^r \frac{x_iy-1}{y-x_i} = \sum_{a=0}^\infty Q'_a(\mathbf{x})y^a. \end{cases} \tag{3.10}$$

We set $Q_a(\mathbf{x})=Q'_a(\mathbf{x})=0$ if $a<0$. For each non-negative integer a, it is not difficult to verify that $Q_a(\mathbf{x})$ (resp. $Q'_a(\mathbf{x})$) is a symmetric polynomial in variables $x_1,x_2,\cdots,x_r$ (resp. $x_1^{-1},\dots,x_r^{-1}$). Further,

$$Q'_a(\mathbf{x}) = Q_a(x_1^{-1},\dots,x_r^{-1}).$$

Lemma 3.11. *Suppose R is an integral domain which contains the identity 1_R, and the units $u_1,u_2,\cdots,u_r,q,(q-q^{-1})$ such that $u_iu_j^{\pm1}\neq 1$. Let F be the quotient field of R. For $a\in\mathbb{Z}$, define*

$$\omega_a = \sum_{i=1}^r (\gamma_d(u_i)+\delta^{-1}\varrho(u_i^2-1)\prod_{j\neq i}u_j)u_i^a \prod_{j\neq i}\frac{u_iu_j-1}{u_i-u_j}$$

where ϱ is defined in (2) of Proposition 3.2 for $d=r$. Suppose that $a\geq 0$. Then

$$\omega_a = \begin{cases} A + \sum\limits_{k=0}^{a-1} \frac{1+(-1)^k}{2} Q_{a-1-k}(\mathbf{u}) - \delta^{-1}\varrho\delta_{a0}, & \text{if } 2\nmid r, \\ A - \sum\limits_{k=0}^{a} \frac{1+(-1)^k}{2} Q_{a-k}(\mathbf{u}) - \delta^{-1}\varrho\delta_{a0}, & \text{if } 2\mid r, \end{cases} \tag{3.12}$$

and for $a>0$,

$$\omega_{-a} = \begin{cases} B + \sum_{k=0}^{a-1} \frac{1+(-1)^k}{2} Q'_{a-1-k}(\mathbf{u}), & \text{if } 2\nmid r, \\ B - \sum_{k=0}^{a-2} \frac{1+(-1)^k}{2} Q'_{a-2-k}(\mathbf{u}), & \text{if } 2\mid r, \end{cases} \tag{3.13}$$

where

$$\begin{cases} A &= \frac{1+(-1)^a}{2} + \delta^{-1}\varrho Q_a(\mathbf{u})\prod_{i=1}^d u_i, \\ B &= \frac{1+(-1)^a}{2} - \delta^{-1}\varrho Q'_a(\mathbf{u})\prod_{i=1}^d u_i. \end{cases}$$

In particular, $\omega_a\in R$ for all $a\in\mathbb{Z}$.

Proof. The result for $a=0$ follows from Proposition 3.2. Suppose a is a positive integer. We define

$$f(z) = \frac{\gamma_d(z)z^a}{z^2-1}\prod_{l=1}^r \frac{u_lz-1}{z-u_l} + \delta^{-1}\varrho z^{a-1}\prod_{l=1}^r \frac{u_l(u_lz-1)}{z-u_l}.$$

Then $\omega_a = \sum_{i=1}^r Res_{z=u_i}f(z)dz$. By residue theorem for complete non-singular curves for $f(z)$,

$$\omega_a = -\sum_{v\in\{\infty,\pm1\}} Res_{z=v}f(z)dz. \tag{3.14}$$

When r is odd, $Res_{z=1}f(z)dz = -\frac{1}{2}$ and $Res_{z=-1}f(z)dz = -\frac{(-1)^a}{2}$. On the other hand, $-z^{-2}f(z^{-1}) = z^{-a}\varphi_1(z)+z^{-a-1}\varphi_2(z)$, where

$$\begin{cases} \varphi_1(z) = (z^2-1)^{-1}\prod_{l=1}^r \frac{z-u_l}{u_lz-1} \\ \varphi_2(z) = -\delta^{-1}\varrho\prod_{l=1}^r \frac{(z-u_l)u_l}{u_lz-1}. \end{cases}$$

Also, we have $(\varphi_2(z))^{(a)}|_{z=0} = -a!\delta^{-1}\varrho\prod_{l=1}^r u_l Q_a(\mathbf{u})$ and

$$\begin{aligned}(\varphi_1(z))^{(a-1)}|_{z=0} &= \sum_{k=0}^{a-1}\binom{a-1}{k}(\frac{1}{z^2-1})^{(k)}\Big(\prod_{l=1}^r\frac{z-u_l}{u_lz-1}\Big)^{(a-1-k)}|_{z=0}\\ &= \sum_{k=0}^{a-1}\binom{a-1}{k}(-1)k!\frac{1+(-1)^k}{2}(a-1-k)!Q_{a-1-k}(\mathbf{u})\\ &= -(a-1)!\sum_{k=0}^{a-1}\frac{1+(-1)^k}{2}Q_{a-1-k}(\mathbf{u}).\end{aligned}$$

Note that $Res_{z=\infty}f(z)dz = Res_{z=0}f_1(z)dz + Res_{z=0}f_2(z)dz$ where $f_1(z) = z^{-a}\varphi_1(z)$ and $f_2(z) = z^{-a-1}\varphi_2(z)$. So,

$$Res_{z=\infty}f(z)dz = -\sum_{k=0}^{a-1}\frac{1+(-1)^k}{2}Q_{a-1-k}(\mathbf{u}) - \delta^{-1}\varrho\prod_{l=1}^r u_lQ_a(\mathbf{u}),$$

and (3.12) follows immediately from (3.14).

When r is even, $\gamma_d(z) = -z$. When we compute $Res_{z=\infty}f(z)dz$, we need compute $(\varphi_1(z))^{(a)}|_{z=0}$. Therefore, we need use a instead of $a-1$ in (3.12) for odd r. This implies the result for (3.12) in the case $2 \mid r$.

One can verify (3.13) by similar arguments as above. Since $Q_a(\mathbf{u})$ (resp. $Q'_a(\mathbf{u})$) are polynomials in variables $u_1, u_2, \dots, u_r$ (resp. $u_1^{-1}, \dots, u_r^{-1}$), $\omega_a \in R$ for all $a \in \mathbb{Z}$. □

Definition 3.15. Suppose R is a commutative ring which contains identity 1_R, and the units $q, (q-q^{-1}), u_i, 1 \le i \le r$. Let $\Omega = \{\omega_a \mid a \in \mathbb{Z}\} \subseteq R$. Then $\Omega \cup \{\varrho\}$ is called $\mathbf{u}$-admissible if ω_a satisfy (3.12), (3.13) for $a \in \mathbb{Z}$ and ϱ is given in Proposition 3.2 for $d = r$.

By Theorem 3.8, $\mathscr{B}_{r,2}$ is a free over R with dimension $3r^2$ if $R = \mathbb{Q}(u_1, u_2, \cdots, u_r, q)$, and $\Omega \cup \{\varrho\}$ is $\mathbf{u}$-admissible. We will show that $\mathscr{B}_{r,n}$ is free over a commutative ring with rank $r^n(2n-1)!!$ if $\Omega\cup\{\varrho\}$ is $\mathbf{u}$-admissible and $2 \nmid r$.

Motivated by Nazarov's work on Brauer algebras in [19], we define two generating functions

$$\begin{cases} \widetilde{w}_{1,+}(y) = \sum_{a=0}^\infty \omega_a y^{-a}\\ \widetilde{w}_{1,-}(y) = \sum\limits_{a=1}^{\infty} \omega_{-a}y^{-a}.\end{cases}$$

Lemma 3.16. *Suppose y is an indeterminant. Then $\Omega\cup\{\varrho\}$ is $\mathbf{u}$-admissible if and only if ϱ is given in Proposition 3.2 for $d = r$ and*

$$\begin{cases}\widetilde{w}_{1,+}(y) &= \frac{y^2}{y^2-1} - \delta^{-1}\varrho + (\delta^{-1}\varrho\prod\limits_{l=1}^r u_l + \frac{y\gamma_d(y)}{y^2-1})\prod_{l=1}^r u_l\prod_{l=1}^r\frac{y-u_l^{-1}}{y-u_l},\\ \widetilde{w}_{1,-}(y) &= \frac{1}{y^2-1} + \delta^{-1}\varrho - \frac{1}{\prod_{l=1}^r u_l}(\delta^{-1}\varrho\prod\limits_{l=1}^r u_l - \frac{y}{\gamma_d(y)(y^2-1)})\prod_{l=1}^r\frac{y-u_l}{y-u_l^{-1}}.\end{cases}$$

Proof. When r is odd, we have $\sum_{a=0}^\infty \frac{1+(-1)^a}{2}y^{-a} = \frac{y^2}{y^2-1}$. By (3.10),

$$\varrho\prod_{l=1}^r u_l\sum_{a=0}^\infty\delta^{-1}Q_a(\mathbf{u})y^{-a} = \delta^{-1}\varrho\prod_{l=1}^r u_l^2\prod_{l=1}^r\frac{y-u_l^{-1}}{y-u_l},$$

and

$$\sum_{a=0}^{\infty}\sum_{k=0}^{a-1}\frac{1+(-1)^k}{2}Q_{a-1-k}(\mathbf{u})y^{-a}=(y^{-1}+y^{-3}+\cdots)\sum_{a=0}^{\infty}Q_a(\mathbf{u})y^{-a}$$
$$=\frac{y}{y^2-1}\prod_{l=1}^{r}u_l\prod_{l=1}^{r}\frac{y-u_l^{-1}}{y-u_l}$$

Taking the sum of the above equalities yields the formula for $\widetilde{w}_{1,+1}(y)$ in the case r is odd. Similarly, one can verify the formulae in other cases. $\square$

Corollary 3.17. *$\Omega\cup\{\rho\}$ is $\mathbf{u}$-admissible if and only if*

a) ϱ is defined by Proposition 3.2 for $d=r$,

b) ω_a for $0\le a\le r-1$ is determined by $\tilde{w}_{1,+}(y)$,

c) Ω is admissible.

Proof. "$\Longrightarrow$" Let $c=\delta^{-1}\varrho u_1\cdots u_r$. By Lemma 3.16, we have

$$(\widetilde{w}_{1,+}(y)-\frac{y^2}{y^2-1}+\delta^{-1}\varrho)(\widetilde{w}_{1,-}(y)-\frac{1}{y^2-1}-\delta^{-1}\varrho)=(\frac{y\gamma_d(y)}{1-y^2}-c)(\frac{y\gamma_d(y)^{-1}}{(1-y^2)}+c).$$

For any positive integers r and all ϱ in Proposition 3.2 for $d=r$, we have

$$(\frac{y\gamma_d(y)}{1-y^2}-c)(\frac{y\gamma_d(y)^{-1}}{(1-y^2)}+c)=\frac{y^2}{(1-y^2)^2}-\delta^{-2}.$$

So,

$$(\widetilde{w}_{1,+}(y)-\frac{y^2}{y^2-1}+\delta^{-1}\varrho)(\widetilde{w}_{1,-}(y)-\frac{1}{y^2-1}-\delta^{-1}\varrho)=\frac{y^2}{(1-y^2)^2}-\delta^{-2}. \tag{3.18}$$

Multiplying $(1-y^2)^2$ on both sides of (4) yields an equality given by Laurent polynomials in indeterminate y. Comparing the coefficients of $y^i, i\le 4$ on both sides of such equality yields the second equality in (2.5). If $\omega_a=\sum_{j=1}^{r}u_j^a\gamma_j$ where γ_j's are given in Proposition 3.2 for $d=r$, then

$$\sum_{a=0}^{r}(-1)^{r-a}\sigma_{r-a}(\mathbf{u})\omega_{a+b}=\sum_{a=0}^{r}(-1)^{r-a}\sigma_{r-a}(\mathbf{u})\sum_{j=1}^{r}u_j^{a+b}\gamma_j$$
$$=\sum_{j=1}^{r}(\sum_{a=0}^{r}(-1)^{r-a}\sigma_{r-a}(\mathbf{u})u_j^a)u_j^b\gamma_j$$
$$=0.$$

So, Ω is admissible.

"$\Longleftarrow$" Conversely, if Ω is admissible, then ω_a for all $a\in\mathbb{Z}$ are determined by $\omega_0,\omega_1,\ldots,\omega_{r-1}$, uniquely. This implies the result. $\square$

4. The seminormal representations of $\mathscr{B}_{r,n}(\mathbf{u})$

In this section, unless otherwise stated, we always keep the following assumptions:

Assumption 4.1. *(1) R is a field which contains non zero elements q, and $u_i, 1\le i\le r$ with $o(q^2)>2n$. (2) The parameters $(u_1,u_2,\ldots,u_r)$ are generic in the sense of Definition 4.5. (3) The root conditions 4.18 hold. (4) $\Omega\cup\{\varrho\}$ is $\mathbf{u}$-admissible.*

The main purpose of this section is to construct the seminormal representations for $\mathscr{B}_{r,n}$ over R. We remark that the method we are using is the same as that for cyclotomic Nazarov-Wenzl algebras in [3]. We start by recalling some combinatorics.

A partition of m is a sequence of non–negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_i \geq \lambda_{i+1}$ for all positive integers i and $|\lambda| := \lambda_1 + \lambda_2 + \cdots = m$. Similarly, an r-partition of m is an ordered r–tuple $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions $\lambda^{(s)}$ such that $|\lambda| := |\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = m$. Let $\Lambda_r^+(m)$ be the set of all r-partitions of m.

If λ and μ are two r-partitions we say that μ is obtained from λ by adding a box if there exists a pair (i,s) such that $\mu_i^{(s)} = \lambda_i^{(s)} + 1$ and $\mu_j^{(t)} = \lambda_j^{(t)}$ for $(j,t) \neq (i,s)$. In this situation we will also say that λ is obtained from μ by removing a box and we write $\lambda \subset \mu$ and $\mu \setminus \lambda = (s, i, \lambda_i^{(s)} + 1)$. We will also say that the triple $(s, i, \lambda_i^{(s)} + 1)$ is an addable (resp. removable) node of λ (resp. μ) which is in the i-th row, $\lambda_i^{(s)} + 1$-column of s-th component of λ (resp. μ).

Fix an integer m with $0 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Let $\lambda \in \Lambda_r^+(n-2m)$. It has been defined in [3] that an n–updown λ–tableau, or more simply an updown λ–tableau, is a sequence $\mathfrak{t} = (\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2, \dots, \mathfrak{t}_n)$ of r-partitions where $\mathfrak{t}_n = \lambda$ and the r-partition $\mathfrak{t}_i$ is obtained from $\mathfrak{t}_{i-1}$ by either *adding* or *removing* a box, for $i = 1, \dots, n$. When $i = 0$, we always assume that $\mathfrak{t}_i = \emptyset$. Let $\mathscr{T}_n^{ud}(\lambda)$ be the set of n–updown λ–tableaux.

There is an equivalence relation $\overset{k}{\sim}$ on $\mathscr{T}_n^{ud}(\lambda)$, which has been defined in [3]. Suppose $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. Then $\mathfrak{t} \overset{k}{\sim} \mathfrak{s}$ if $\mathfrak{t}_j = \mathfrak{s}_j$ whenever $1 \leq j \leq n$ and $j \neq k$, for $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$. The following result has been proved in [3].

Lemma 4.2. *Suppose $s \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$. Then there is a bijection between the set of all addable and removable nodes of $\mathfrak{s}_{k-1}$ and the set of $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{t} \overset{k}{\sim} \mathfrak{s}$.*

Suppose that $\mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ where $\lambda \in \Lambda_r^+(n-2f)$ and $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. For each positive integer $k \leq n$, either $\mathfrak{t}_k \subset \mathfrak{t}_{k-1}$ or $\mathfrak{t}_{k-1} \subset \mathfrak{t}_k$. We define

$$c_{\mathfrak{t}}(k) = \begin{cases} u_s q^{2(j-i)}, & \text{if } \mathfrak{t}_k \backslash \mathfrak{t}_{k-1} = (i,j,s), \\ u_s^{-1} q^{-2(j-i)}, & \text{if } \mathfrak{t}_{k-1} \backslash \mathfrak{t}_k = (i,j,s). \end{cases} \tag{4.3}$$

We call $c_{\mathfrak{t}}(k)$ the content of k in $\mathfrak{t}$. Let $\alpha = (i,j,s)$. We also define

$$c_\lambda(\alpha) = \begin{cases} u_s q^{2(j-i)}, & \text{if } \alpha \text{ is an addable node of } \lambda, \\ u_s^{-1} q^{-2(j-i)}, & \text{if } \alpha \text{ is a removable node of } \lambda. \end{cases} \tag{4.4}$$

We write $c(\alpha)$ instead of $c_\lambda(\alpha)$ if there is no confusion.

The following condition, which is a counterpart of the generic condition for cyclotomic Nazarov-Wenzl algebras in [3], guarantees the existence of the seminormal representations for $\mathscr{B}_{r,n}$.

Definition 4.5. The parameters $\mathbf{u} = (u_1, \dots, u_r)$ are **generic** for $\mathscr{B}_{r,n}$ if whenever there exists $d \in \mathbb{Z}$ such that either $u_i u_j^{\pm 1} = q^{2d} 1_R$ and $i \neq j$, or $u_i = \pm q^d \cdot 1_R$ then $|d| \geq 2n$.

Lemma 4.6. *Suppose that the parameters $\mathbf{u}$ are generic for $\mathscr{B}_{r,n}$ and that $o(q^2) > 2n$. Let $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ where $\lambda \in \Lambda_r^+(n-2f)$ and $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Then*

- *a) $\mathfrak{s} = \mathfrak{t}$ if and only if $c_{\mathfrak{s}}(k) = c_{\mathfrak{t}}(k)$, for all positive integers $k \leq n$.*
- *b) $c_{\mathfrak{s}}(k) \neq c_{\mathfrak{s}}(k+1)$, for all positive integers $k < n$.*
- *c) if $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$ then $c_{\mathfrak{s}}(k) \neq c_{\mathfrak{t}}(k)^{\pm 1}$, whenever $\mathfrak{t} \overset{k}{\sim} \mathfrak{s}$ and $\mathfrak{t} \neq \mathfrak{s}$,*
- *d) $c_{\mathfrak{t}}(k) \neq \pm q^{\pm 1}$ for all positive integers $k \leq n$.*

Proof. (a)-(c) can be proved by the arguments similar to those in [3]. The key point is that the assumptions imply that the contents of the addable and removable nodes in λ are distinct so an up-down λ–tableau $\mathfrak{s}$ is uniquely determined by the sequence of contents $c_{\mathfrak{s}}(k)$, for $k = 1, \dots, n$. (d) can be verified by direct computation. □

Unless otherwise stated, we fix a $\lambda \in \Lambda_r^+(n-2f)$. Motivated by Ariki, Mathas and Rui's work on cyclotomic Nazarov-Wenzl algebras in [3], we introduce the following rational functions in an indeterminate y. Such functions will play a key role in the construction of seminormal representations of $\mathscr{B}_{r,n}$.

Definition 4.7. Suppose that $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$. For $1 \le k \le n$, define rational functions $W_k(y, \mathfrak{s})$ by

$$W_k(y,\mathfrak{s}) = \frac{y^2}{y^2-1} - \delta^{-1}\varrho + (\delta^{-1}\varrho \prod_{i=1}^r u_i + \frac{y\gamma_r(y)}{y^2-1}) \prod_{i=1}^r u_i \prod_\alpha \frac{y - c^{-1}(\alpha)}{y - c(\alpha)},$$

where α runs over the addable and removable nodes of the r-partition $\mathfrak{s}_{k-1}$.

Lemma 4.8. *Suppose λ is an r-partition. Then*

$$\prod_\alpha c(\alpha) = \prod_{i=1}^r u_i \tag{4.9}$$

where α runs over all addable nodes and removable nodes of λ.

Proof. It is known that the number of addable nodes of a partition, say μ, is equal to the number of the removable nodes of μ plus 1. We arrange the removable nodes (resp. addable nodes) of μ from top to bottom. Therefore, we assume that $(p, r_i, s_i), 1 \le i \le k$ (resp. $(p, a_i, b_i), 1 \le i \le k+1$) are all removable (resp. addable) nodes of $\lambda^{(p)}$, the p-th component of λ. We have $a_1 = 1$, $b_{k+1} = 1$, $a_i = r_{i-1} + 1$ and $s_j = b_j - 1$ for $2 \le i \le k+1$ and $1 \le j \le k$. By (4.4),

$$\prod_\alpha c(\alpha) = u_p q^{2(\sum_{i=1}^{k+1}(b_i - a_i) + \sum_{i=1}^k (r_i - s_i))} = u_p.$$

Multiplying the previous equality for all positive integers $p \le r$ yields (4.9). □

Lemma 4.10. *Suppose that $\mathbf{u}$ is generic and $o(q^2) > 2n$. Let $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ and $1 \le k \le n$. Then*

$$\frac{W_k(y,\mathfrak{s})}{y} = \sum_\alpha \Big(\operatorname*{Res}_{y=c(\alpha)} \frac{W_k(y,\mathfrak{s})}{y} \Big) \cdot \frac{1}{y - c(\alpha)},$$

where α runs over the addable and removable nodes of $\mathfrak{s}_{k-1}$.

Proof. Since $\mathbf{u}$ is generic and $o(q^2) > 2n$, $c(\alpha)$ are pairwise distinct for different addable and removable nodes α of λ. Further, we have $c(\alpha) \notin \{0, \pm 1\}$. Therefore, we can write

$$\frac{W_k(y,\mathfrak{s})}{y} = a + \frac{b}{y} + \frac{c}{y-1} + \frac{d}{y+1} + \sum_\alpha \Big(\operatorname*{Res}_{y=c(\alpha)} \frac{W_k(y,\mathfrak{s})}{y} \Big) \cdot \frac{1}{y - c(\alpha)},$$

for some $a, b, c, d \in R$, where α runs over the addable and removable nodes of $\mathfrak{s}_{k-1}$. In order to prove the result, we need verify $a = b = c = d = 0$. In fact, $a = \lim_{y\to\infty} \frac{W_k(y,\mathfrak{s})}{y} = 0$. We have $b = \operatorname{Res}_{y=0} \frac{W_k(y,\mathfrak{s})}{y} = -\delta^{-1}\varrho + \delta^{-1}\varrho \prod_{i=1}^r u_i^2 \prod_\alpha c(\alpha)^{-2} \overset{(4.9)}{=} -\delta^{-1}\varrho + \delta^{-1}\varrho = 0$. One can verify $c = d = 0$ similarly. □

The following definition is the same as those for cyclotomic Nazarov-Wenzl algebras if we use our rational functions $W_k(y, \mathfrak{s})$ instead of those for cyclotomic Nazarov-Wenzl algebras in [3].

Definition 4.11. Let $\lambda \in \Lambda_r^+(n-2f)$ for some non-negative integer $f \le \lfloor \frac{n}{2} \rfloor$. Assume that k is a positive integer with $k \le n$. If $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$, then we define the scalars $E_{\mathfrak{s}\mathfrak{t}}(k) \in R$ by

$$E_{\mathfrak{s}\mathfrak{t}}(k) = \begin{cases} \operatorname*{Res}_{y=c_{\mathfrak{s}}(k)} \dfrac{W_k(y,\mathfrak{s})}{y}, & \text{if } \mathfrak{s} = \mathfrak{t}, \\ \sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)}\sqrt{E_{\mathfrak{t}\mathfrak{t}}(k)}, & \text{if } \mathfrak{s} \ne \mathfrak{t} \text{ and } \mathfrak{t} \overset{k}{\sim} \mathfrak{s}, \\ 0, & \text{otherwise.} \end{cases}$$

We remark that we have to fix the choice of square roots $\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)}$, for $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ and $1 \le k \le n$, which we will illustrate late.

In [3], there is no definition for $E_{\mathfrak{s}\mathfrak{s}}(k)$ under the assumption $\mathfrak{s}_{k-1} \ne \mathfrak{s}_{k+1}$. In the current paper, we do not need such a definition either.

If r is odd, then $\rho^{-1} \in \{u_1 \cdots u_r, -u_1 \cdots u_r\}$. It follows from Definition 4.7 that

$$(4.12)\quad E_{\mathfrak{s}\mathfrak{s}}(k) = \begin{cases} \frac{1}{\varrho c_{\mathfrak{s}}(k)}\big(\frac{c_{\mathfrak{s}}(k)-c_{\mathfrak{s}}(k)^{-1}}{\delta}+1\big)\prod_\alpha \frac{c_{\mathfrak{s}}(k)-c(\alpha)^{-1}}{c_{\mathfrak{s}}(k)-c(\alpha)}, & \text{if } \varrho^{-1} = u_1\cdots u_r, \\ \frac{1}{\varrho c_{\mathfrak{s}}(k)}\big(\frac{c_{\mathfrak{s}}(k)-c_{\mathfrak{s}}(k)^{-1}}{\delta}-1\big)\prod_\alpha \frac{c_{\mathfrak{s}}(k)-c(\alpha)^{-1}}{c_{\mathfrak{s}}(k)-c(\alpha)}, & \text{if } \varrho^{-1} = -u_1\cdots u_r, \end{cases}$$

where α runs over all addable and removable nodes of $\mathfrak{s}_{k-1}$ with $\alpha \ne \mathfrak{s}_k \setminus \mathfrak{s}_{k-1}$.

If r is even, then $\varrho^{-1} \in \{q^{-1}u_1\cdots u_r, -qu_1\cdots u_r\}$. So,

$$(4.13)\qquad E_{\mathfrak{s}\mathfrak{s}}(k) = \begin{cases} \frac{1}{\varrho\delta}\big(1-\frac{q^2}{c_{\mathfrak{s}}(k)^2})\big)\prod_\alpha \frac{c_{\mathfrak{s}}(k)-c(\alpha)^{-1}}{c_{\mathfrak{s}}(k)-c(\alpha)}, & \text{if } \varrho^{-1} = q^{-1}\prod_{i=1}^r u_i, \\ \frac{1}{\varrho\delta}\big(1-\frac{1}{q^2 c_{\mathfrak{s}}(k)^2}\big)\prod_\alpha \frac{c_{\mathfrak{s}}(k)-c(\alpha)^{-1}}{c_{\mathfrak{s}}(k)-c(\alpha)}, & \text{if } \varrho^{-1} = -q\prod_{i=1}^r u_i, \end{cases}$$

where α runs over all addable and removable nodes of $\mathfrak{s}_{k-1}$ with $\alpha \ne \mathfrak{s}_k \setminus \mathfrak{s}_{k-1}$.

It follows from Lemma 4.6 and (4.12)–(4.13) that

$$(4.14)\qquad E_{\mathfrak{s}\mathfrak{t}}(k) \ne 0, \text{ if } \mathfrak{s} \overset{k}{\sim} \mathfrak{t}.$$

Rewriting Lemma 4.10 yields the following equality:

$$(4.15)\qquad \frac{W_k(y,\mathfrak{s})}{y} = \sum_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s}} \frac{E_{\mathfrak{t}\mathfrak{t}}(k)}{y-c_{\mathfrak{t}}(k)}.$$

Given two partitions $\mathfrak{s}$ and $\mathfrak{t}$ write $\mathfrak{s} \ominus \mathfrak{t} = \alpha$ if either $\mathfrak{t} \subset \mathfrak{s}$ and $\mathfrak{s} \setminus \mathfrak{t} = \alpha$, or $\mathfrak{s} \subset \mathfrak{t}$ and $\mathfrak{t} \setminus \mathfrak{s} = \alpha$.

Let $\mathfrak{S}_n$ be the symmetric group in n letters. As an Coxeter group, $\mathfrak{S}_n$ is generated by $s_i := (i, i+1)$ subject to the relations

$$\begin{cases} s_i^2 = 1, & \text{if } 1 \le i \le n-1, \\ s_i s_j = s_j s_i & \text{if } |i-j| > 1 \\ s_i s_j s_i = s_j s_i s_j, & \text{if } |i-j| = 1. \end{cases}$$

Let $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1} \ne \mathfrak{s}_{k+1}$, for some k, $1 \le k < n$. Suppose that $\mathfrak{s}_k \ominus \mathfrak{s}_{k-1}$ and $\mathfrak{s}_{k+1} \ominus \mathfrak{s}_k$ are in different rows and in different columns. It is defined in [3] that

$$s_k\mathfrak{s} = (\mathfrak{s}_1, \cdots, \mathfrak{s}_{k-1}, \mathfrak{t}_k, \mathfrak{s}_{k+1}, \cdots, \mathfrak{s}_n) \in \mathscr{T}_n^{ud}(\lambda)$$

where $\mathfrak{t}_k$ is the r-partition which is uniquely determined by the conditions $\mathfrak{t}_k \ominus \mathfrak{s}_{k+1} = \mathfrak{s}_{k-1} \ominus \mathfrak{s}_k$ and $\mathfrak{s}_{k-1} \ominus \mathfrak{t}_k = \mathfrak{s}_k \ominus \mathfrak{s}_{k+1}$. If the nodes $\mathfrak{s}_k \ominus \mathfrak{s}_{k-1}$ and $\mathfrak{s}_{k+1} \ominus \mathfrak{s}_k$ are either in the same row, or in the same column, then $s_k\mathfrak{s}$ is not defined.

Definition 4.16. Let $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1} \ne \mathfrak{s}_{k+1}$, for some k, $1 \le k < n$. We define

$$a_{\mathfrak{s}}(k) = \frac{\delta c_{\mathfrak{s}}(k+1)}{c_{\mathfrak{s}}(k+1) - c_{\mathfrak{s}}(k)} \quad \text{and} \quad b_{\mathfrak{s}}(k) = \sqrt{1 - a_{\mathfrak{s}}(k)^2 + \delta a_{\mathfrak{s}}(k)}.$$

We will fix the choice of square root for $b_{\mathfrak{s}}(k)$ in (4.18). Since $\mathbf{u}$ is generic, by Lemma 4.6(b), $c_{\mathfrak{s}}(k+1) - c_{\mathfrak{s}}(k) \neq 0$. So, the formula for $a_s(k)$ makes sense.

As in [3], we do not define $a_{\mathfrak{s}}(k)$ and $b_{\mathfrak{s}}(k)$ when $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$. The following result can be verified easily.

Lemma 4.17. *Suppose that $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ and $1 \le k < n$. Then:*

a) If $s_k\mathfrak{s}$ is defined then $c_{\mathfrak{s}}(k) = c_{s_k\mathfrak{s}}(k+1)$ and $c_{\mathfrak{s}}(k+1) = c_{s_k\mathfrak{s}}(k)$; consequently, $a_{s_k\mathfrak{s}}(k) = \delta - a_{\mathfrak{s}}(k)$.

b) If $s_k\mathfrak{s}$ is not defined then $a_{\mathfrak{s}}(k) \in \{q, -q^{-1}\}$ and $b_{\mathfrak{s}}(k) = 0$.

Finally, if $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$ and $\mathfrak{t} \overset{k}{\sim} \mathfrak{s}$, where $1 \le k < n$, we set

$$T_{\mathfrak{s}\mathfrak{t}}(k) = \delta \frac{E_{\mathfrak{s}\mathfrak{t}}(k) - \delta_{\mathfrak{s}\mathfrak{t}}}{c_{\mathfrak{s}}(k)c_{\mathfrak{t}}(k) - 1}.$$

Note that $c_{\mathfrak{s}}(k)c_{\mathfrak{t}}(k) \neq 1$ by Lemma 4.6.

We will assume that we have chosen the square roots in the definitions of $b_{\mathfrak{s}}(k)$ and $E_{\mathfrak{s}\mathfrak{t}}(k)$ so that the following equalities hold.

Assumption 4.18 (Root conditions)**.** *We assume that the ring R is large enough so that $\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)} \in R$ and $b_{\mathfrak{s}}(k) = \sqrt{1 - a_{\mathfrak{s}}(k)^2 + \delta a_{\mathfrak{s}}(k)} \in R$, for all $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}_n^{ud}(\lambda)$ and $1 \le k < n$, and that the following equalities hold:*

a) If $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$ and $s_k\mathfrak{s}$ is defined then $b_{s_k\mathfrak{s}}(k) = b_{\mathfrak{s}}(k)$.

b) If $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$ and $\mathfrak{s} \overset{l}{\sim} \mathfrak{t}$, where $|k - l| > 1$, then $b_{\mathfrak{s}}(k) = b_{\mathfrak{t}}(k)$.

c) If $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$, $\mathfrak{s}_k \neq \mathfrak{s}_{k+2}$ and $s_k\mathfrak{s}$ and $s_{k+1}\mathfrak{s}$ are both defined then $b_{s_{k+1}\mathfrak{s}}(k) = b_{s_k\mathfrak{s}}(k+1)$.

d) If $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$ and $\mathfrak{s}_k = \mathfrak{s}_{k+2}$ then $\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)}\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k+1)} = 1$.

e) If $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$, $\mathfrak{t}_{k-1} = \mathfrak{t}_{k+1}$ and $E_{\mathfrak{s}\mathfrak{s}}(k) = E_{\mathfrak{t}\mathfrak{t}}(k)$ then $\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)} = \sqrt{E_{\mathfrak{t}\mathfrak{t}}(k)}$.

f) If $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$, $\mathfrak{s}_k = \mathfrak{s}_{k+2}$ and $\mathfrak{t} \overset{k+1}{\sim} \mathfrak{s}$, $\mathfrak{u} \overset{k}{\sim} \mathfrak{s}$ with $s_k\mathfrak{t}$ and $s_{k+1}\mathfrak{u}$ both defined and $s_k\mathfrak{t} = s_{k+1}\mathfrak{u}$ then $b_{\mathfrak{t}}(k)\sqrt{E_{\mathfrak{t}\mathfrak{t}}(k+1)} = b_{\mathfrak{u}}(k+1)\sqrt{E_{\mathfrak{u}\mathfrak{u}}(k)}$.

The following is the main result of this section.

Theorem 4.19. *Let $\mathscr{B}_{r,n}$ be over a field R such that the Assumption 4.1 holds. Let $\Delta(\lambda)$ be the R–vector space with basis $\{ v_{\mathfrak{s}} \mid \mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda) \}$. Then $\Delta(\lambda)$ becomes a $\mathscr{B}_{r,n}$-module via*

- $$T_k v_{\mathfrak{s}} = \begin{cases} \sum_{\mathfrak{t} \overset{k}{\sim} \mathfrak{s}} T_{\mathfrak{s}\mathfrak{t}}(k) v_{\mathfrak{t}}, & \text{if } \mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}, \\ a_{\mathfrak{s}}(k)v_{\mathfrak{s}} + b_{\mathfrak{s}}(k)v_{s_k\mathfrak{s}}, & \text{if } \mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}, \end{cases}$$
- $$E_k v_{\mathfrak{s}} = \begin{cases} \sum_{\mathfrak{t} \overset{k}{\sim} \mathfrak{s}} E_{\mathfrak{s}\mathfrak{t}}(k) v_{\mathfrak{t}}, & \text{if } \mathfrak{s}_{k-1} = \mathfrak{s}_{k+1} \\ 0, & \text{if } \mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}, \end{cases}$$
- $X_i v_{\mathfrak{s}} = c_{\mathfrak{s}}(i) v_{\mathfrak{s}}$,

for $1 \le k < n$ and $1 \le i \le n$. When $s_k\mathfrak{s}$ is not defined, we set $v_{s_k\mathfrak{s}} = 0$.

Definition 4.20. We call $\Delta(\lambda)$ the seminormal representation of $\mathscr{B}_{r,n}(\mathbf{u})$ with respect to λ for $\lambda \in \Lambda_r^+(n - 2f)$, and $0 \le f \le \lfloor \frac{n}{2} \rfloor$.

In the remainder of this section, we will prove Theorem 4.19. The rational functions $W_k(y, \mathfrak{s})$ play the key role. As in [3], we will work with formal (infinite) linear combinations of elements of $\Delta(\lambda)$ and $\mathscr{B}_{r,n}$. Let $Z(A)$ be the center of an algebra A.

Lemma 4.21. *Suppose that R is a commutative ring which contains 1 and invertible elements $q, \delta, u_1, \dots, u_r$ such that $\Omega \cup \{\varrho\}$ is $\mathbf{u}$-admissible. Given two integers $k \geq 1$ and $a \geq 0$. Then there is a $\omega_k^{(a)} \in Z(\mathscr{B}_{r,k-1}) \cap R[X_1^{\pm 1}, \dots, X_{k-1}^{\pm 1}]$ such that*

$$E_k X_k^a E_k = \omega_k^{(a)} E_k.$$

Moreover, the generating series $\widetilde{W}_k(y) = \sum_{a=0}^{\infty} \omega_k^{(a)} y^{-a}$ satisfies

$$\frac{\widetilde{W}_{k+1}(y) + \delta^{-1}\varrho - \frac{y^2}{y^2-1}}{\widetilde{W}_k(y) + \delta^{-1}\varrho - \frac{y^2}{y^2-1}} = \frac{(y - X_k)^2}{(y - X_k^{-1})^2} \cdot \frac{y - q^{-2}X_k^{-1}}{y - q^{-2}X_k} \cdot \frac{y - q^2 X_k^{-1}}{y - q^2 X_k}. \tag{4.22}$$

Proof. We prove the result by induction on k. When $k = 1$, the result follows from Definition 2.1(g). In this case, $\tilde{W}_1(y) = \tilde{w}_{1,+}(y)$. Suppose that we have already proved the result for all positive integers which are less than $k+1$. In order to prove the result for $k+1$, we start with the identity

$$T_k \frac{1}{y - X_k} = \frac{1}{y - X_{k+1}} T_k + \delta \frac{1}{yX_k - 1} E_k \frac{1}{y - X_k} - \frac{\delta X_{k+1}}{(y - X_k)(y - X_{k+1})}, \tag{4.23}$$

We multiply $(y - X_{k+1})(yX_k - 1)$ (resp. $(y - X_k)$) on the left (resp. right) of (4.23). Then we use Definition 2.1(f),(j) to get the identity. Multiplying (4.23) on the left by E_k and replacing y by y^{-1} yields

$$E_k \frac{1}{y - X_k} T_k = E_k\left(\frac{\varrho X_k}{yX_k - 1} - \frac{\delta}{(y - X_k)(yX_k - 1)}\right) + \delta \frac{\widetilde{W}_k(y)}{y} E_k \frac{1}{yX_k - 1} \tag{4.24}$$

Multiplying T_k on the right of (4.23) and using (4.23)–(4.24), we have

$$\begin{aligned}
T_k \frac{1}{y - X_k} T_k = & \frac{1}{y - X_{k+1}} + \delta T_k \frac{1}{y - X_k} + \frac{\delta^2 X_{k+1}}{(y - X_k)(y - X_{k+1})} \\
& - \frac{\delta^2}{yX_k - 1} E_k \frac{1}{y - X_k} - \frac{\varrho \delta X_k}{yX_k - 1} E_k - \frac{\delta^2 X_{k+1}^2}{(y - X_k)^2 (y - X_{k+1})} \\
& + \frac{\delta^2}{X_k (y - X_k)(yX_k - 1)} E_k \frac{1}{y - X_k} - \frac{\delta}{y - X_k} T_k \frac{X_k}{y - X_k} \\
& - \frac{\delta^2 X_{k+1}}{(y - X_k)^2} + \frac{\delta^2}{X_k (y - X_k)} E_k \frac{1}{y - X_k} + \frac{\varrho \delta}{yX_k - 1} E_k \frac{X_k}{yX_k - 1} \\
& - \frac{\delta^2}{yX_k - 1} E_k \frac{1}{(y - X_k)(yX_k - 1)} + \frac{\delta^2 \widetilde{W}_k(y)}{y(yX_k - 1)} E_k \frac{1}{yX_k - 1}
\end{aligned}$$

Note that

$$\begin{aligned}
& \frac{\delta^2 X_{k+1}}{(y - X_k)(y - X_{k+1})} - \frac{\delta^2 X_{k+1}^2}{(y - X_k)^2 (y - X_{k+1})} - \frac{\delta^2 X_{k+1}}{(y - X_k)^2} \\
= & \frac{\delta^2 X_k}{(y - X_k)^2} - \frac{\delta^2 X_k}{(y - X_k)^2} \frac{y}{y - X_{k+1}}.
\end{aligned}$$

By our induction assumption,

$$\begin{aligned}
E_{k+1}T_k\frac{1}{y-X_k}T_kE_{k+1} = & (\frac{1}{y}-\frac{\delta^2 X_k}{(y-X_k)^2})E_{k+1}\frac{y}{y-X_{k+1}}E_{k+1} \\
& +(\frac{\varrho^{-1}\delta}{y-X_k}+\frac{\delta^2\omega_0X_k}{(y-X_k)^2}-\frac{\delta^2}{(yX_k-1)(y-X_k)} \\
& -\frac{\varrho\delta X_k}{yX_k-1}+\frac{\delta^2}{X_k(y-X_k)^2}+\frac{\delta^2}{X_k(y-X_k)^2(yX_k-1)} \\
& -\frac{\varrho^{-1}\delta X_k}{(y-X_k)^2}+\frac{\varrho\delta X_k}{(yX_k-1)^2}-\frac{\delta^2}{(yX_k-1)^2(y-X_k)} \\
& +\frac{\delta^2\widetilde{W}_k(y)}{y(yX_k-1)^2})E_{k+1}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& E_{k+1}T_k\frac{1}{y-X_k}T_kE_{k+1} = E_{k+1}E_kT_{k+1}^{-1}\frac{1}{y-X_k}T_kE_{k+1} \\
= & E_{k+1}E_k\frac{1}{y-X_k}E_kE_{k+1}-\delta E_{k+1}E_k\frac{1}{y-X_k}T_kE_{k+1}+\delta\varrho^{-1}E_{k+1}\frac{1}{y-X_k}
\end{aligned}$$

We use (4.24) to compute the second term in the right hand side of the above equality. Thus,

$$\begin{aligned}
E_{k+1}T_k\frac{1}{y-X_k}T_kE_{k+1} = & (\frac{\widetilde{W}_k(y)}{y}+\frac{\varrho^{-1}\delta}{y-X_k}-\frac{\varrho\delta X_k}{yX_k-1} \\
& +\frac{\delta^2}{(yX_k-1)(y-X_k)}-\frac{\delta^2\widetilde{W}_k(y)}{y(yX_k-1)})E_{k+1}
\end{aligned}$$

Comparing the first and third expressions of $E_{k+1}T_k\frac{1}{y-X_k}T_kE_{k+1}$ yields

$$\begin{aligned}
& (\frac{1}{y}-\frac{\delta^2 X_k}{(y-X_k)^2})E_{k+1}\frac{y}{y-X_{k+1}}E_{k+1}+(\frac{\varrho^{-1}\delta}{y-X_k}+\frac{\delta^2\omega_0X_k}{(y-X_k)^2} \\
& -\frac{\delta^2}{(yX_k-1)(y-X_k)}-\frac{\varrho\delta X_k}{yX_k-1}+\frac{\delta^2}{X_k(y-X_k)^2} \\
& +\frac{\delta^2}{X_k(y-X_k)^2(yX_k-1)}-\frac{\varrho^{-1}\delta X_k}{(y-X_k)^2}+\frac{\varrho\delta X_k}{(yX_k-1)^2} \\
& -\frac{\delta^2}{(yX_k-1)^2(y-X_k)}+\frac{\delta^2\widetilde{W}_k(y)}{y(yX_k-1)^2})E_{k+1} \\
& =(\frac{\widetilde{W}_k(y)}{y}+\frac{\varrho^{-1}\delta}{y-X_k}-\frac{\varrho\delta X_k}{yX_k-1}+\frac{\delta^2}{(yX_k-1)(y-X_k)}-\frac{\delta^2\widetilde{W}_k(y)}{y(yX_k-1)})E_{k+1}
\end{aligned}$$

So,

$$
\begin{aligned}
&(\frac{1}{y}-\frac{\delta^2X_k}{(y-X_k)^2})E_{k+1}\frac{y}{y-X_{k+1}}E_{k+1}\\
=&(-\frac{\varrho^{-1}\delta}{y-X_k}-\frac{\delta^2\omega_0X_k}{(y-X_k)^2}\\
&+\frac{\delta^2}{(yX_k-1)(y-X_k)}+\frac{\varrho\delta X_k}{yX_k-1}-\frac{\delta^2}{X_k(y-X_k)^2}\\
&-\frac{\delta^2}{X_k(y-X_k)^2(yX_k-1)}+\frac{\varrho^{-1}\delta X_k}{(y-X_k)^2}-\frac{\varrho\delta X_k}{(yX_k-1)^2}\\
&+\frac{\delta^2}{(yX_k-1)^2(y-X_k)}-\frac{\delta^2\widetilde{W}_k(y)}{y(yX_k-1)^2}\\
&+\frac{\widetilde{W}_k(y)}{y}+\frac{\varrho^{-1}\delta}{y-X_k}-\frac{\varrho\delta X_k}{yX_k-1}+\frac{\delta^2}{(yX_k-1)(y-X_k)}-\frac{\delta^2\widetilde{W}_k(y)}{y(yX_k-1)})E_{k+1}\\
=&(\frac{1}{y}-\frac{\delta^2X_k}{(yX_k-1)^2})\widetilde{W}_k(y)E_{k+1}+\varrho\delta X_k(\frac{1}{(y-X_k)^2}-\frac{1}{(yX_k-1)^2})E_{k+1}\\
&+\frac{\delta^2X_ky^2(1-X_k^2)}{(yX_k-1)^2(y-X_k)^2}E_{k+1}
\end{aligned}
$$

Multiplying $(yX_k-1)^2(y-X_k)^2$ on both sides of above equation yields

$$
\begin{aligned}
&(yX_k-1)^2(\frac{(y-X_k)^2}{y}-\delta^2X_k)E_{k+1}\frac{y}{y-X_{k+1}}E_{k+1}\\
=&(y-X_k)^2(\frac{(yX_k-1)^2}{y}-\delta^2X_k)\widetilde{W}_k(y)E_{k+1}+\varrho\delta X_k((yX_k-1)^2\\
&-(y-X_k)^2)E_{k+1}+\delta^2X_ky^2(1-X_k^2)E_{k+1}.
\end{aligned}
$$

So,

$$
\begin{aligned}
&(y-X_k^{-1})^2((y-X_k)^2-\delta^2X_ky)E_{k+1}\frac{y}{y-X_{k+1}}E_{k+1}\\
=&(y-X_k)^2((y-X_k^{-1})^2-\delta^2X_k^{-1}y)\widetilde{W}_k(y)E_{k+1}+\delta^2X_k^{-1}y^3(1-X_k^2)E_{k+1}\\
&+\varrho\delta y(X_k(y-X_k^{-1})^2-X_k^{-1}(y-X_k)^2)E_{k+1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(y-X_k^{-1})^2(y-q^{-2}X_k)(y-q^2X_k)E_{k+1}\frac{y}{y-X_{k+1}}E_{k+1}\\
=&(y-X_k)^2(y-q^{-2}X_k^{-1})(y-q^2X_k^{-1})\widetilde{W}_k(y)E_{K+1}\\
&-(\delta^{-1}\varrho-\frac{y^2}{y^2-1})((y-X_k^{-1})^2(y-q^{-2}X_k)(y-q^2X_k)\\
&-(y-X_k)^2(y-q^{-2}X_k^{-1})(y-q^2X_k^{-1}))
\end{aligned}
$$

Thus,

$$E_{k+1}\frac{y}{y-X_{k+1}}E_{k+1}=\widetilde{W}_{k+1}(y)E_{k+1},$$

where $\widetilde{W}_{k+1}(y)$ satisfies the recursive relation (4.22).

Note that $X_kT_j=T_jX_k$ and $X_kE_j=E_jX_k$ for all positive integers $j\le k-2$. By induction assumption and (4.22), $\omega_{k+1}^{(a)}$ commutates with $E_1,\dots,E_{k-2}$ and $T_1,\dots,T_{k-2}$. In order to verify

$$\omega_{k+1}^{(a)}Y=Y\omega_{k+1}^{(a)},\ \text{for } Y\in\{T_{k-1},E_{k-1}\}, \tag{4.25}$$

we use the following series

$$\sum_{m\geq 0} a_m z^m = \frac{(1+X_{k-1}z)(1+X_kz)}{(1+X_{k-1}^{-1}z)(1+X_k^{-1}z)},$$

where $z\in\{-y^{-1},-q^{\pm 2}y\}$,

$$\begin{aligned}
&a_0=1, a_1=X_{k-1}+X_k-X_{k-1}^{-1}-X_k^{-1},\\
&a_2=X_kX_{k-1}-X_k^{-1}X_{k-1}^{-1}-(X_k^{-1}+X_{k-1}^{-1})(X_k+X_{k-1}-X_k^{-1}-X_{k-1}^{-1})\\
&a_m=-(X_{k-1}^{-1}+X_k^{-1})a_{m-1}-X_{k-1}^{-1}X_k^{-1}a_{m-2},\ \text{for } m\geq 3.
\end{aligned}$$

By induction, we have $E_{k-1}a_j=a_jE_{k-1}=0$ for all non-negative integers j.

We verify $T_{k-1}a_m=a_mT_{k-1}$ for all $m\geq 0$ by induction on m. There is nothing to be proved when $m=0$ since $a_0=1$. By Definition 2.1(f),

$$\begin{cases} T_{k-1}(X_{k-1}+X_k)=(X_{k-1}+X_k)T_{k-1}+\delta(X_kE_{k-1}-E_{k-1}X_k)\\ T_{k-1}(X_{k-1}^{-1}+X_k^{-1})=(X_{k-1}^{-1}+X_k^{-1})T_{k-1}+\delta(X_{k-1}^{-1}E_{k-1}-E_{k-1}X_{k-1}^{-1})\end{cases}$$

So, $T_{k-1}a_1=a_1T_{k-1}$. When $m\geq 2$, since T_{k-1} commute with X_kX_{k-1} and $X_k^{-1}X_{k-1}^{-1}$, we have, by induction assumption that

$$\begin{aligned}
T_{k-1}a_m-a_mT_{k-1}=&(X_{k-1}^{-1}+X_k^{-1})a_{m-1}T_{k-1}-T_{k-1}(X_{k-1}^{-1}+X_k^{-1})a_{m-1}\\
=&(X_{k-1}^{-1}+X_k^{-1})a_{m-1}T_{k-1}-(X_{k-1}^{-1}+X_k^{-1})T_{k-1}a_{m-1}\\
&-\delta(X_{k-1}^{-1}E_{k-1}-E_{k-1}X_{k-1}^{-1})a_{m-1}=0.
\end{aligned}$$

Using (4.22) twice, we have

$$\begin{aligned}
\frac{\widetilde{W}_{k+1}(y)+\delta^{-1}\varrho-\frac{y^2}{y^2-1}}{\widetilde{W}_{k-1}(y)+\delta^{-1}\varrho-\frac{y^2}{y^2-1}}=&\frac{(y-X_k)^2}{(y-X_k^{-1})^2}\cdot\frac{y-q^{-2}X_k^{-1}}{y-q^{-2}X_k}\cdot\frac{y-q^2X_k^{-1}}{y-q^2X_k}\\
&\frac{(y-X_{k-1})^2}{(y-X_{k-1}^{-1})^2}\cdot\frac{y-q^{-2}X_{k-1}^{-1}}{y-q^{-2}X_{k-1}}\cdot\frac{y-q^2X_{k-1}^{-1}}{y-q^2X_{k-1}}.
\end{aligned}\tag{4.26}$$

Thus, T_{k-1},E_{k-1} commute with the right hand side of (4.26). By induction assumption, T_{k-1},E_{k-1} commutes with $\tilde{W}_{k-1}(y)$, it has to commute with $\tilde{W}_{k+1}(y)$. Thus, $\omega_{k+1}^{(a)}$ commutes with T_{k-1} and E_{k-1}.

Now, we prove $\omega_k^{(a)}\in R[X_1^{\pm 1},\cdots,X_{k-1}^{\pm 1}]$. Let $g_k(X_k)=(y-X_k^{-1})^2(y-q^{-2}X_k)(y-q^2X_k)$. By (4.22),

$$\widetilde{W}_{k+1}(y)g_k(X_k)=\widetilde{W}_k(y)g_k(X_k^{-1})+(X_k-X_k^{-1})\delta^2y((\delta^{-1}\rho-1)y^2-\delta^{-1}\rho).$$

Comparing the coefficients of y^j for $j\leq 4$ on both sides of the last equation, we have the following results:

a) $j=4$: $\omega_{k+1}^{(0)}=\omega_k^{(0)}$. By induction assumption, $\omega_{k+1}^{(0)}=\omega_0$.

b) $j=3$: $\omega_{k+1}^{(1)}=\omega_k^{(1)}+\delta\rho^{-1}(X_k-X_k^{-1})$.

c) $j=2$:

$$\omega_{k+1}^{(2)}=\omega_k^{(2)}-\omega_k^{(1)}(2X_k+(q^2+q^{-2})X_k^{-1})+\omega_{k+1}^{(1)}(2X_k^{-1}+(q^2+q^{-2})X_k).$$

d) $j=1$:

$$\begin{aligned}
&\omega_{k+1}^{(3)}-\omega_k^{(3)}+(\omega_{k+1}^{(1)}-\omega_k^{(1)})(X_k^2+X_k^{-2}+2(q^2+q^{-2}))\\
=&(\omega_{k+1}^{(2)}-\omega_k^{(0)})(2X_k^{-1}+(q^2+q^{-2})X_k)-\delta\rho(X_k-X_k^{-1})\\
&+(\omega_{k+1}^{(0)}-\omega_k^{(2)})(2X_k+(q^2+q^{-2})X_k^{-1}).
\end{aligned}$$

e) $j \le 0$:

$$
\begin{aligned}
&\omega_{k+1}^{(-j+4)} + \omega_{k+1}^{(-j)} + (\omega_{k+1}^{(-j+2)} - \omega_k^{(-j+2)})(X_k^2 + X_k^{-2} + 2(q^2+q^{-2}))\\
=&\omega_k^{(-j)} + \omega_k^{(-j+4)} + (\omega_{k+1}^{(-j+1)} - \omega_k^{(-j+3)})(2X_k + (q^2+q^{-2})X_k^{-1})\\
&+ (\omega_{k+1}^{(-j+3)} - \omega_k^{(-j+1)})(2X_K^{-1} + (q^2+q^{-2})X_k).
\end{aligned}
$$

By induction assumptions on k and a together with the formulae in (a)-(e), we have that $\omega_k^{(a)} \in R[X_1^{\pm1}, \cdots, X_{k-1}^{\pm1}]$. □

Corollary 4.27. *Suppose that R is a commutative ring which contains 1 and invertible elements $q, \delta, u_1, \ldots, u_r$ such that $\Omega \cup \{\varrho\}$ is $\mathbf{u}$-admissible. Given a positive integer $k \le n-1$. If $a \in \mathbb{Z}$, then*

a) $E_k X_k^a T_{k-1}^{\pm1} E_k = \sum_{i=-a}^{a} f_i X_{k-1}^i E_k$;

b) $E_k X_k^a T_{k+1}^{\pm1} E_k = \sum_{i=-a}^{a} g_i X_{k+2}^i E_k$,

where $f_i, g_i \in Z(\mathscr{B}_{r,k-1}) \cap R[X_1^{\pm1}, \ldots, X_{k-1}^{\pm1}]$ for $-a \le i \le a$.

Proof. Both (a) an (b) follow from Lemma 2.3(1)(2)(4)(5) and Lemma 4.21 and Definition 2.1. □

By Lemma 4.21, we have

$$
\frac{\widetilde{W}_k(y) + \delta^{-1}\varrho - \frac{y^2}{y^2-1}}{\widetilde{W}_1(y) + \delta^{-1}\varrho - \frac{y^2}{y^2-1}} = \prod_{i=1}^{k-1} \frac{(y-X_i)^2}{(y-X_i^{-1})^2} \cdot \frac{y - q^{-2}X_i^{-1}}{y - q^{-2}X_i} \cdot \frac{y - q^2 X_i^{-1}}{y - q^2 X_i}. \tag{4.28}
$$

Since $\omega_k^{(a)} \in R[X_1^{\pm1}, \cdots, X_{k-1}^{\pm1}]$, we can define $\widetilde{W}_k(y, \mathfrak{s}) \in R((y^{-1}))$ by

$$
\widetilde{W}_k(y) v_{\mathfrak{s}} = \widetilde{W}_k(y, \mathfrak{s}) v_{\mathfrak{s}}.
$$

Proposition 4.29. *Given an $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ and a positive integer $k \le n$, we have $W_k(y, \mathfrak{s}) = \widetilde{W}_k(y, \mathfrak{s})$.*

Proof. As $\Omega \cup \{\varrho\}$ is $\mathbf{u}$-admissible, by Lemma 3.16 and (4.28), we have

$$
\begin{aligned}
&\widetilde{W}_k(y, \mathfrak{s}) + \delta^{-1}\varrho - \frac{y^2}{y^2-1}\\
=&A \cdot \prod_{\ell=1}^{r} \frac{(y - u_\ell^{-1})}{(y - u_\ell)} \times \prod_{i=1}^{k-1} \frac{(y - c_{\mathfrak{s}}(i))^2}{(y - c_{\mathfrak{s}}(i)^{-1})^2} \cdot \frac{y - q^{-2}c_{\mathfrak{s}}(i)^{-1}}{y - q^{-2}c_{\mathfrak{s}}(i)} \cdot \frac{y - q^2 c_{\mathfrak{s}}(i)^{-1}}{y - q^2 c_{\mathfrak{s}}(i)}.
\end{aligned}
$$

where

$$
A = (\varrho\delta^{-1} \prod_{\ell=1}^{r} u_\ell + \frac{y\gamma_r(y)}{y^2-1}) \prod_{\ell=1}^{r} u_\ell.
$$

By arguments similar to those for [3, 4.17] we can verify that

$$
\widetilde{W}_k(y, \mathfrak{s}) + \delta^{-1}\varrho - \frac{y^2}{y^2-1} = A \prod_{\alpha} \frac{y - c(\alpha)^{-1}}{y - c(\alpha)},
$$

where α runs over the addable and removable nodes of $\mathfrak{s}_{k-1}$. This proves $W_k(y, \mathfrak{s}) = \tilde{W}_k(y, \mathfrak{s})$. □

One can verify the following result by similar arguments to those for [3, 4.18].

Corollary 4.30. *Suppose that $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ and that $1 \le k < n$ and $a \ge 0$. Then $E_k X_k^a E_k v_{\mathfrak{s}} = \omega_k^{(a)} E_k v_{\mathfrak{s}}$.*

Lemma 4.31. *Suppose that $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$ and $\mathfrak{s}_k = \mathfrak{s}_{k+2}$. Then $E_{\mathfrak{s}\mathfrak{s}}(k) E_{\mathfrak{s}\mathfrak{s}}(k+1) = 1$.*

Proof. By (4.22) and Proposition 4.29,

$$W_{k+1}(y,\mathfrak{s})+\delta^{-1}\varrho-\frac{y^2}{y^2-1}=(W_k(y,\mathfrak{s})+\delta^{-1}\varrho-\frac{y^2}{y^2-1})\\ \times\frac{(y-c_\mathfrak{s}(k))^2}{(y-c_\mathfrak{s}(k)^{-1})^2}\frac{(y-q^{-2}c_\mathfrak{s}(k)^{-1})}{(y-q^{-2}c_\mathfrak{s}(k))}\frac{(y-q^2c_\mathfrak{s}(k)^{-1})}{(y-q^2c_\mathfrak{s}(k))},$$

where $W_k(y,\mathfrak{s})$ is given by Definition 4.7. Note that $c_\mathfrak{s}(k)c_\mathfrak{s}(k+1)=1$ and

$$E_{\mathfrak{s}\mathfrak{s}}(k+1)=Res_{y=c_\mathfrak{s}(k+1)}\frac{W_{k+1}(y,\mathfrak{s})}{y}=Res_{y=c_\mathfrak{s}(k)^{-1}}\frac{W_{k+1}(y,\mathfrak{s})}{y}.$$

There are four cases we need to discuss:

Case 1. $2\nmid r$ **and** $\varrho^{-1}=u_1u_2\cdots u_r$:

$$E_{\mathfrak{s}\mathfrak{s}}(k+1)=\Big(\delta^{-1}c_\mathfrak{s}(k)+\frac{c_\mathfrak{s}(k)^2}{1-c_\mathfrak{s}(k)^2}\Big)\varrho^{-1}\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_\mathfrak{s}(k)^{-1}-c_\mathfrak{t}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-c_\mathfrak{t}(k)}\\ \Big(c_\mathfrak{s}(k)^{-1}-c_\mathfrak{s}(k)\Big)\frac{c_\mathfrak{s}(k)^{-1}-q^{-2}c_\mathfrak{s}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-q^{-2}c_\mathfrak{s}(k)}\frac{c_\mathfrak{s}(k)^{-1}-q^{2}c_\mathfrak{s}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-q^{2}c_\mathfrak{s}(k)}\\ =\frac{\varrho\delta c_\mathfrak{s}(k)^2}{c_\mathfrak{s}(k)^2+\delta c_\mathfrak{s}(k)-1}\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_\mathfrak{s}(k)-c_\mathfrak{t}(k)}{c_\mathfrak{s}(k)-c_\mathfrak{t}(k)^{-1}}=\frac{1}{E_{\mathfrak{s}\mathfrak{s}}(k)}$$

Case 2. $2\nmid r$ **and** $\varrho^{-1}=-\prod_{l=1}^r u_i$:

$$E_{\mathfrak{s}\mathfrak{s}}(k+1)=\Big(-\delta^{-1}c_\mathfrak{s}(k)+\frac{c_\mathfrak{s}(k)^2}{1-c_\mathfrak{s}(k)^2}\Big)\Big(-\varrho^{-1}\Big)\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_\mathfrak{s}(k)^{-1}-c_\mathfrak{t}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-c_\mathfrak{t}(k)}\\ \Big(c_\mathfrak{s}(k)^{-1}-c_\mathfrak{s}(k)\Big)\frac{c_\mathfrak{s}(k)^{-1}-q^{-2}c_\mathfrak{s}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-q^{-2}c_\mathfrak{s}(k)}\frac{c_\mathfrak{s}(k)^{-1}-q^{2}c_\mathfrak{s}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-q^{2}c_\mathfrak{s}(k)}\\ =\frac{\varrho\delta c_\mathfrak{s}(k)^2}{c_\mathfrak{s}(k)^2-\delta c_\mathfrak{s}(k)-1}\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_\mathfrak{s}(k)-c_\mathfrak{t}(k)}{c_\mathfrak{s}(k)-c_\mathfrak{t}(k)^{-1}}=\frac{1}{E_{\mathfrak{s}\mathfrak{s}}(k)}$$

Case 3. $2\mid r$ **and** $\varrho^{-1}=q^{-1}\prod_{l=1}^r u_i$:

$$E_{\mathfrak{s}\mathfrak{s}}(k+1)=\Big(q\delta^{-1}c_\mathfrak{s}(k)-\frac{c_\mathfrak{s}(k)}{1-c_\mathfrak{s}(k)^2}\Big)q\varrho^{-1}\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_\mathfrak{s}(k)^{-1}-c_\mathfrak{t}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-c_\mathfrak{t}(k)}\\ \Big(c_\mathfrak{s}(k)^{-1}-c_\mathfrak{s}(k)\Big)\frac{c_\mathfrak{s}(k)^{-1}-q^{-2}c_\mathfrak{s}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-q^{-2}c_\mathfrak{s}(k)}\frac{c_\mathfrak{s}(k)^{-1}-q^{2}c_\mathfrak{s}(k)^{-1}}{c_\mathfrak{s}(k)^{-1}-q^{2}c_\mathfrak{s}(k)}\\ =\frac{\varrho\delta c_\mathfrak{s}(k)^2}{c_\mathfrak{s}(k)^2-q^2}\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_\mathfrak{s}(k)-c_\mathfrak{t}(k)}{c_\mathfrak{s}(k)-c_\mathfrak{t}(k)^{-1}}=\frac{1}{E_{\mathfrak{s}\mathfrak{s}}(k)}$$

Case 4. $2\mid r$ **and** $\varrho^{-1}=-q\prod_{l=1}^r u_i$:

$$\begin{aligned}E_{\mathfrak{s}\mathfrak{s}}(k+1) =&\Big(-q^{-1}\delta^{-1}c_{\mathfrak{s}}(k)-\frac{c_{\mathfrak{s}}(k)}{1-c_{\mathfrak{s}}(k)^2}\Big)\Big(-q^{-1}\varrho^{-1}\Big)\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_{\mathfrak{s}}(k)^{-1}-c_{\mathfrak{t}}(k)^{-1}}{c_{\mathfrak{s}}(k)^{-1}-c_{\mathfrak{t}}(k)}\\ &\Big(c_{\mathfrak{s}}(k)^{-1}-c_{\mathfrak{s}}(k)\Big)\frac{c_{\mathfrak{s}}(k)^{-1}-q^{-2}c_{\mathfrak{s}}(k)^{-1}}{c_{\mathfrak{s}}(k)^{-1}-q^{-2}c_{\mathfrak{s}}(k)}\frac{c_{\mathfrak{s}}(k)^{-1}-q^{2}c_{\mathfrak{s}}(k)^{-1}}{c_{\mathfrak{s}}(k)^{-1}-q^{2}c_{\mathfrak{s}}(k)}\\ =&\frac{\varrho\delta c_{\mathfrak{s}}(k)^2}{c_{\mathfrak{s}}(k)^2-q^{-2}}\prod_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s},\mathfrak{t}\neq\mathfrak{s}}\frac{c_{\mathfrak{s}}(k)-c_{\mathfrak{t}}(k)}{c_{\mathfrak{s}}(k)-c_{\mathfrak{t}}(k)^{-1}}=\frac{1}{E_{\mathfrak{s}\mathfrak{s}}(k)}.\end{aligned}$$

We remark that we use Lemma 4.2, (4.9) and (4.12)–(4.13) when we verify the equalities in cases 1–4. □

The following result can be proved by similar arguments to those for [3, 4.20]. The only difference is that we use our rational functions $W_k(y,\mathfrak{s})$ instead of those for cyclotomic Nazarov-Wenzl algebras.

Lemma 4.32. *Fix an integer k with $1\le k<n-1$ and suppose that $\mathfrak{s},\mathfrak{t},\mathfrak{u}\in\mathscr{T}_n^{ud}(\lambda)$ such that $\mathfrak{s}_{k-1}=\mathfrak{s}_{k+1}$, $\mathfrak{s}_k=\mathfrak{s}_{k+2}$, $\mathfrak{t}\overset{k+1}{\sim}\mathfrak{s}$, $\mathfrak{u}\overset{k}{\sim}\mathfrak{s}$ and that $s_k\mathfrak{t}$ and $s_{k+1}\mathfrak{u}$ are both defined with $s_k\mathfrak{t}=s_{k+1}\mathfrak{u}$. Then*

$$b_{\mathfrak{t}}(k)^2E_{\mathfrak{t}\mathfrak{t}}(k+1)=b_{\mathfrak{u}}(k+1)^2E_{\mathfrak{u}\mathfrak{u}}(k).$$

The following combinatorial identities will be used in the proof of Theorem 4.19.

Proposition 4.33. *Suppose that $\mathfrak{s},\mathfrak{t}'\in\mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1}=\mathfrak{s}_{k+1}$, $\mathfrak{s}_k\neq\mathfrak{s}_{k+2}$, $\mathfrak{t}'\overset{k}{\sim}\mathfrak{s}$ and $\mathfrak{t}'\neq\mathfrak{s}$, where $1\le k<n-1$. Then the following identities hold:*

a) $\displaystyle\sum_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s}}\frac{E_{\mathfrak{t}\mathfrak{t}}(k)}{c_{\mathfrak{s}}(k)c_{\mathfrak{t}}(k)-1}=\delta^{-1}\varrho+\frac{1}{c_{\mathfrak{s}}(k)^2-1}$,

b) $\displaystyle\sum_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s}}\frac{E_{\mathfrak{t}\mathfrak{t}}(k)}{(c_{\mathfrak{s}}(k)c_{\mathfrak{t}}(k)-1)^2}=\frac{c_{\mathfrak{s}}(k)^2+1}{(c_{\mathfrak{s}}(k)^2-1)^2}-\delta^{-1}\varrho+(\frac{1}{\delta^2}-\frac{c_{\mathfrak{s}}(k)^2}{(c_{\mathfrak{s}}(k)^2-1)^2})\frac{1}{E_{\mathfrak{s}\mathfrak{s}}(k)}$

c) $\displaystyle\sum_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s}}\frac{E_{\mathfrak{t}\mathfrak{t}}(k)}{(c_{\mathfrak{s}}(k)c_{\mathfrak{t}}(k)-1)(c_{\mathfrak{t}}(k)c_{\mathfrak{t}'}(k)-1)}=\frac{c_{\mathfrak{s}}(k)c_{\mathfrak{t}'}(k)+1}{(c_{\mathfrak{s}}(k)^2-1)(c_{\mathfrak{t}'}(k)^2-1)}-\delta^{-1}\varrho$,

Proof. Evaluating both sides of (4.15) at $y=c_{\mathfrak{s}}(k)^{-1}$ and using (4.7) gives (a). By Proposition 4.29 and Corollary 4.30 we have

$$E_k\frac{1}{(y-X_k)(v-X_k)}E_kv_{\mathfrak{s}}=\frac{1}{v-y}\Big(\frac{W_k(y,\mathfrak{s})}{y}-\frac{W_k(v,\mathfrak{s})}{v}\Big)E_kv_{\mathfrak{s}}.$$

Comparing the coefficients of $v_{\mathfrak{s}}$ on both sides of this equation yields

$$\sum_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s}}\frac{E_{\mathfrak{t}\mathfrak{t}}(k)}{(y-c_{\mathfrak{t}}(k))(v-c_{\mathfrak{t}}(k))}=\frac{1}{v-y}\Big\{\frac{W_k(y,\mathfrak{s})}{y}-\frac{W_k(v,\mathfrak{s})}{v}\Big\}.$$

Let $y=c_{\mathfrak{s}}(k)^{-1}$. We use (a) to rewrite the above equality and obtain the following equality:

$$\begin{aligned}&\sum_{\mathfrak{t}\overset{k}{\sim}\mathfrak{s}}\frac{E_{\mathfrak{t}\mathfrak{t}}(k)}{(1-c_{\mathfrak{s}}(k)c_{\mathfrak{t}}(k))(v-c_{\mathfrak{t}}(k))}\\ &=\frac{W_k(v,\mathfrak{s})+\delta^{-1}\varrho-\frac{v^2}{v^2-1}}{v(1-vc_{\mathfrak{s}}(k))}-v^{-1}\delta^{-1}\varrho+\frac{v+c_{\mathfrak{s}}(k)}{(v^2-1)(1-c_{\mathfrak{s}}(k)^2)}\end{aligned}\tag{4.34}$$

Setting $v=c_{\mathfrak{t}'}(k)^{-1}$ gives (c). Now we set $v=c_{\mathfrak{s}}(k)^{-1}$. There are four cases we need to discuss.

When $2 \nmid r$ and $\prod_{l=1}^r u_i = \varrho^{-1}$, it follows from (4.34) that

$$\begin{aligned}
&\sum_{\mathfrak{t} \overset{k}{\sim} \mathfrak{s}} \frac{E_{\mathfrak{t}\mathfrak{t}}(k)}{(1-c_{\mathfrak{s}}(k)c_{\mathfrak{t}}(k))^2} + \delta^{-1}\varrho - \frac{1+c_{\mathfrak{s}}(k)^2}{(1-c_{\mathfrak{s}}(k)^2)^2} \\
=&\varrho^{-1}\Big(\delta^{-1}c_{\mathfrak{s}}(k) + \frac{c_{\mathfrak{s}}(k)^2}{1-c_{\mathfrak{s}}(k)^2}\Big) \prod_{\substack{\mathfrak{t} \overset{k}{\sim} \mathfrak{s} \\ \mathfrak{t} \neq \mathfrak{s}}} \frac{c_{\mathfrak{s}}(k)^{-1} - c_{\mathfrak{t}}(k)^{-1}}{c_{\mathfrak{s}}(k)^{-1} - c_{\mathfrak{t}}(k)} \frac{-c_{\mathfrak{s}}(k)^{-2}}{c_{\mathfrak{s}}(k)^{-1} - c_{\mathfrak{s}}(k)} \\
=&\frac{c_{\mathfrak{s}}(k)^2 - \delta c_{\mathfrak{s}}(k) - 1}{\delta(c_{\mathfrak{s}}(k)^2-1)^2} \varrho^{-1} c_{\mathfrak{s}}(k)^2 \prod_{\mathfrak{t} \overset{k}{\sim} \mathfrak{s}} c_{\mathfrak{t}}(k)^{-2} \prod_{\mathfrak{t} \neq \mathfrak{s}} \frac{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)}{c_{\mathfrak{t}}(k)^{-1} - c_{\mathfrak{s}}(k)} \\
=&\frac{c_{\mathfrak{s}}(k)^2 - \delta c_{\mathfrak{s}}(k) - 1}{\delta(c_{\mathfrak{s}}(k)^2-1)^2} \varrho c_{\mathfrak{s}}(k)^2 \prod_{\mathfrak{t} \neq \mathfrak{s}} \frac{c_{\mathfrak{t}}(k) - c_{\mathfrak{s}}(k)}{c_{\mathfrak{t}}(k)^{-1} - c_{\mathfrak{s}}(k)} \text{ by (4.9)} \\
=&(\delta^{-2} - \frac{c_{\mathfrak{s}}(k)^2}{(1-c_{\mathfrak{s}}(k)^2)^2}) \frac{1}{E_{\mathfrak{s}\mathfrak{s}}(k)} \text{ by (4.12).}
\end{aligned}$$

This proves (b) under the assumption $2 \nmid r$ and $\prod_{l=1}^r u_i = \varrho^{-1}$. One can verify (b) in other cases. □

We are going to check that the action of $\mathscr{B}_{r,n}(\mathbf{u})$ on $\Delta(\lambda)$ respects the defining relations of $\mathscr{B}_{r,n}(\mathbf{u})$.

Lemma 4.35. *Suppose $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$. Then*

a) $E_i^2 v_{\mathfrak{s}} = \omega_0 E_i v_{\mathfrak{s}}$, *for* $1 \le i < n$.
b) $E_1 X_1^k E_1 v_{\mathfrak{s}} = \omega_k E_1 v_{\mathfrak{s}}$, *for* $k \in \mathbb{Z}$.
c) $(X_1 - u_1)(X_1 - u_2)\cdots(X_1 - u_r)v_{\mathfrak{s}} = 0$.
d) $X_i X_j v_{\mathfrak{s}} = X_j X_i v_{\mathfrak{s}}$ *for* $1 \le i, j \le n$.
e) $E_i X_i X_{i+1} v_{\mathfrak{s}} = X_i X_{i+1} E_i v_{\mathfrak{s}} = E_i v_{\mathfrak{s}}$, $1 \le i \le n-1$.
f) $(T_i X_i - X_{i+1} T_i)v_{\mathfrak{s}} = \delta X_{i+1}(E_i - 1)v_{\mathfrak{s}}$, *for* $1 \le i \le n-1$
g) $(X_i T_i - T_{i+1} X_i)v_{\mathfrak{s}} = \delta(E_i - 1)X_{i+1} v_{\mathfrak{s}}$, *for* $1 \le i \le n-1$
h) $T_k X_l v_{\mathfrak{s}} = X_l T_k v_{\mathfrak{s}}$ *if* $l \ne k, k+1$.
i) $E_k E_{k\pm1} E_k v_{\mathfrak{s}} = E_k v_{\mathfrak{s}}$.
j) $E_k T_k v_{\mathfrak{s}} = \varrho E_k v_{\mathfrak{s}} = T_k E_k v_{\mathfrak{s}}$.
k) $T_i T_j v_{\mathfrak{s}} = T_j T_i v_{\mathfrak{s}}$ *if* $|i-j| > 1$.
l) $X_i X_i^{-1} = X_i^{-1} X_i = 1$ *for* $1 \le i \le n$.

Proof. We have already proved (a) and (b) for $k > 0$ in Corollary 4.30. (c)-(h) and (l) can be verified easily. By (c), we have (b) for all $k \in \mathbb{Z}$ with $k < 0$. (i)-(k) can be proved by arguments similar to those in [3, 4.23,4.25, 4.27a]. When we prove (j) we need to use Proposition 4.33(a) instead of [3, 4.21a]. □

It remains to check the defining relations (b), (c)(ii), (h)(ii) in Definition 2.1.

Lemma 4.36. *Suppose that $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$. Then $(T_k^2 - \delta T_k + \delta\varrho E_k)v_{\mathfrak{s}} = v_{\mathfrak{s}}$.*

Proof. We prove the result by computing the coefficient of $v_{\mathfrak{t}}$ in the expression of $(T_k^2 - \delta T_k + \delta\varrho E_k)v_{\mathfrak{s}}$. There are two cases we have to discuss as follows.

Case 1. $\mathfrak{s}_{k-1} \ne \mathfrak{s}_{k+1}$:
Then $E_k v_{\mathfrak{s}} = 0$. If $s_k\mathfrak{s}$ is not defined then $a_{\mathfrak{s}}(k) \in \{q, -q^{-1}\}$ and $b_{\mathfrak{s}}(k) = 0$ (see Lemma 4.17). So, $(T_k^2 - \delta T_k + \delta\varrho E_k)v_{\mathfrak{s}} = v_{\mathfrak{s}}$. If $s_k\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ then by the choice

of the square roots in (4.18)(a) we have

$$\begin{aligned}(T_k^2-\delta T_k+\delta\varrho E_k)v_{\mathfrak s}=&(T_k-\delta)\Big(a_{\mathfrak s}(k)v_{\mathfrak s}+b_{\mathfrak s}(k)v_{s_k\mathfrak s}\Big)\\ =&a_{\mathfrak s}(k)\Big(a_{\mathfrak s}(k)v_{\mathfrak s}+b_{\mathfrak s}(k)v_{s_k\mathfrak s}\Big)-\delta\Big(a_{\mathfrak s}(k)v_{\mathfrak s}+b_{\mathfrak s}(k)v_{s_k\mathfrak s}\Big)\\ &+b_{\mathfrak s}(k)\Big(a_{s_k\mathfrak s}(k)v_{s_k\mathfrak s}+b_{s_k\mathfrak s}(k)v_{\mathfrak s}\Big)\\ =&v_{\mathfrak s}\quad\text{(by Lemma 4.17.)}\end{aligned}$$

Case 2. $\mathfrak s_{k-1}=\mathfrak s_{k+1}$:

We have

$$\begin{aligned}&(T_k^2-\delta T_k+\delta\varrho E_k)v_{\mathfrak s}\\ =&\sum_{\mathfrak t\overset{k}{\sim}\mathfrak s}\Big(\sum_{\mathfrak v\overset{k}{\sim}\mathfrak s}T_{\mathfrak s\mathfrak v}(k)T_{\mathfrak v\mathfrak t}(k)\Big)v_{\mathfrak t}-\delta\sum_{\mathfrak t\overset{k}{\sim}\mathfrak s}T_{\mathfrak s\mathfrak t}(k)v_{\mathfrak t}+\varrho\delta\sum_{\mathfrak t\overset{k}{\sim}\mathfrak s}E_{\mathfrak s\mathfrak t}(k)v_{\mathfrak t}.\end{aligned}$$

The coefficient of $v_{\mathfrak s}$ in $(T_k^2-\delta T_k+\delta\varrho E_k)v_{\mathfrak s}$ is equal to 1 since

$$\begin{aligned}&\sum_{\mathfrak v\overset{k}{\sim}\mathfrak s}T_{\mathfrak s\mathfrak v}(k)T_{\mathfrak v\mathfrak s}(k)-\delta T_{\mathfrak s\mathfrak s}(k)+\varrho\delta E_{\mathfrak s\mathfrak s}(k)\\ =&\sum_{\mathfrak v\overset{k}{\sim}\mathfrak s}\frac{\delta^2E_{\mathfrak s\mathfrak s}(k)E_{\mathfrak v\mathfrak v}(k)}{(c_{\mathfrak s}(k)c_{\mathfrak v}(k)-1)^2}-\frac{\delta^2(E_{\mathfrak s\mathfrak s}(k)-1)}{c_{\mathfrak s}(k)^2-1}+\delta\varrho E_{\mathfrak s\mathfrak s}(k)+\frac{\delta^2(1-2E_{\mathfrak s\mathfrak s}(k))}{(c_{\mathfrak s}(k)^2-1)^2}\\ =&\delta^2E_{\mathfrak s\mathfrak s}(k)\Big(\frac{1+c_{\mathfrak s}(k)^2}{(1-c_{\mathfrak s}(k)^2)^2}+\frac{1}{\delta^2E_{\mathfrak s\mathfrak s}(k)}-\frac{c_{\mathfrak s}(k)^2}{(1-c_{\mathfrak s}(k)^2)^2E_{\mathfrak s\mathfrak s}(k)}\Big)\\ &+\frac{\delta^2(1-2E_{\mathfrak s\mathfrak s}(k))}{(1-c_{\mathfrak s}(k)^2)^2}+\frac{\delta^2(E_{\mathfrak s\mathfrak s}(k)-1)}{1-c_{\mathfrak s}(k)^2}\quad\text{(by Proposition 4.33(b))}\\ =&1.\end{aligned}$$

If $\mathfrak t\overset{k}{\sim}\mathfrak s$ and $\mathfrak t\neq\mathfrak s$ then the coefficient of $v_{\mathfrak t}$ in $(T_k^2-\delta T_k+\delta\varrho E_k)v_{\mathfrak s}$ is zero since

$$\begin{aligned}&\sum_{\mathfrak v\overset{k}{\sim}\mathfrak s}T_{\mathfrak s\mathfrak v}(k)T_{\mathfrak v\mathfrak t}(k)-\delta T_{\mathfrak s\mathfrak t}(k)+\delta\varrho E_{\mathfrak s\mathfrak t}(k)\\ =&\sum_{\substack{\mathfrak v\sim\mathfrak s\\ \mathfrak s\neq\mathfrak v\neq\mathfrak t}}\frac{\delta^2E_{\mathfrak s\mathfrak v}(k)E_{\mathfrak v\mathfrak t}(k)}{(c_{\mathfrak s}(k)c_{\mathfrak v}(k)-1)(c_{\mathfrak v}(k)c_{\mathfrak t}(k)-1)}+\frac{\delta(E_{\mathfrak s\mathfrak s}(k)-1)}{c_{\mathfrak s}(k)^2-1}\frac{\delta E_{\mathfrak s\mathfrak t}(k)}{c_{\mathfrak s}(k)c_{\mathfrak t}(k)-1}\\ &+\frac{\delta E_{\mathfrak s\mathfrak t}(k)}{c_{\mathfrak s}(k)c_{\mathfrak t}(k)-1}\frac{\delta(E_{\mathfrak t\mathfrak t}(k)-1)}{c_{\mathfrak t}(k)^2-1}-\frac{\delta^2E_{\mathfrak s\mathfrak t}(k)}{c_{\mathfrak s}(k)c_{\mathfrak t}(k)-1}+\delta\varrho E_{\mathfrak s\mathfrak t}(k)\\ =&\delta^2E_{\mathfrak s\mathfrak t}(k)\Big(\sum_{\mathfrak v\sim\mathfrak s}\frac{E_{\mathfrak v\mathfrak v}(k)}{(c_{\mathfrak s}(k)c_{\mathfrak v}(k)-1)(c_{\mathfrak v}(k)c_{\mathfrak t}(k)-1)}-\frac{1}{(c_{\mathfrak s}(k)^2-1)(c_{\mathfrak s}(k)c_{\mathfrak t}(k)-1)}\\ &-\frac{1}{(c_{\mathfrak t}(k)^2-1)(c_{\mathfrak s}(k)c_{\mathfrak t}(k)-1)}-\frac{1}{c_{\mathfrak s}(k)c_{\mathfrak t}(k)-1}+\delta^{-1}\varrho\Big)\\ =&\delta^2E_{\mathfrak s\mathfrak t}(k)\Big(\sum_{\mathfrak v\sim\mathfrak s}\frac{E_{\mathfrak v\mathfrak v}(k)}{(c_{\mathfrak s}(k)c_{\mathfrak v}(k)-1)(c_{\mathfrak v}(k)c_{\mathfrak t}(k)-1)}-\frac{c_{\mathfrak s}(k)c_{\mathfrak t}(k)+1}{(c_{\mathfrak s}(k)^2-1)(c_{\mathfrak t}(k)^2-1)}+\delta^{-1}\varrho\Big)\\ =&0\quad\text{(by Proposition 4.33(c))}.\end{aligned}$$

Therefore, $(T_k^2-\delta T_k+\delta\varrho E_k)v_{\mathfrak s}=v_{\mathfrak s}$. ∎

Proposition 4.37. *Suppose that* $\mathfrak s\in\mathscr T_n^{ud}(\lambda)$. *Then*

a) $E_{k+1}E_kv_{\mathfrak s}=T_kT_{k+1}E_kv_{\mathfrak s}$.

b) $E_{k+1}E_kv_{\mathfrak s}=E_{k+1}T_kT_{k+1}v_{\mathfrak s}$.

Proof. (a) We assume that $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$ since otherwise $E_{k+1}E_k v_{\mathfrak{s}} = T_k T_{k+1} E_k v_{\mathfrak{s}} = 0$. Let $\tilde{\mathfrak{s}} \in \mathscr{T}_n^{ud}(\lambda)$ be such that $\tilde{\mathfrak{s}} \overset{k}{\sim} \mathfrak{s}$ and $\tilde{\mathfrak{s}}_k = \mathfrak{s}_{k+2}$. Note that $E_k v_{\mathfrak{s}} = 0$ for any $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ with $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$. So,

$$\begin{aligned}
&T_k E_{k+1} E_k v_{\mathfrak{s}} - \delta E_{k+1} E_k v_{\mathfrak{s}} + \delta E_k v_{\mathfrak{s}} \\
=&(T_k - \delta) E_{k+1} E_{\mathfrak{s},\tilde{\mathfrak{s}}}(k) v_{\tilde{\mathfrak{s}}} + \delta \sum_{\mathfrak{t} \overset{k}{\sim} \mathfrak{s}} E_{\mathfrak{s}\mathfrak{t}}(k) v_{\mathfrak{t}} \\
=&(T_k - \delta) E_{\mathfrak{s},\tilde{\mathfrak{s}}}(k) \sum_{\mathfrak{t} \overset{k+1}{\sim} \tilde{\mathfrak{s}}} E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1) v_{\mathfrak{t}} + \delta \sum_{\mathfrak{t} \overset{k}{\sim} \mathfrak{s}} E_{\mathfrak{s}\mathfrak{t}}(k) v_{\mathfrak{t}} \\
=&E_{\mathfrak{s},\tilde{\mathfrak{s}}}(k) E_{\tilde{\mathfrak{s}},\tilde{\mathfrak{s}}}(k+1) \sum_{\mathfrak{t} \overset{k}{\sim} \tilde{\mathfrak{s}}} T_{\tilde{\mathfrak{s}}\mathfrak{t}}(k) v_{\mathfrak{t}} + \sum_{\mathfrak{t} \overset{k+1}{\sim} \tilde{\mathfrak{s}}, \mathfrak{t} \neq \tilde{\mathfrak{s}}} E_{\mathfrak{s},\tilde{\mathfrak{s}}}(k) E_{\tilde{\mathfrak{s}},\mathfrak{t}}(k+1) (a_{\mathfrak{t}}(k) v_{\mathfrak{t}} + b_{\mathfrak{t}}(k) v_{s_k \mathfrak{t}}) \\
&- \delta E_{\mathfrak{s},\tilde{\mathfrak{s}}}(k) \sum_{\mathfrak{t} \overset{k+1}{\sim} \tilde{\mathfrak{s}}} E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1) v_{\mathfrak{t}} + \delta \sum_{\mathfrak{t} \overset{k}{\sim} \mathfrak{s}} E_{\mathfrak{s}\mathfrak{t}}(k) v_{\mathfrak{t}}.
\end{aligned}$$

If $s_k \mathfrak{t}$ is defined, for $\mathfrak{t}$ in the second sum, then $(s_k \mathfrak{t})_k \neq \mathfrak{s}_{k+2}$ and $\mathfrak{u} = s_{k+1} s_k \mathfrak{t}$ is also defined. Further, we have $\mathfrak{u} \overset{k}{\sim} \tilde{\mathfrak{s}}$ and $\mathfrak{u} \neq \mathfrak{s}$. Similarly,

$$T_{k+1} E_k v_{\mathfrak{s}} = E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k) \sum_{\mathfrak{t} \overset{k+1}{\sim} \tilde{\mathfrak{s}}} T_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1) v_{\mathfrak{t}} + \sum_{\mathfrak{t} \overset{k}{\sim} \tilde{\mathfrak{s}}, \mathfrak{t} \neq \tilde{\mathfrak{s}}} E_{\mathfrak{s}\mathfrak{t}}(k) \Big(a_{\mathfrak{t}}(k+1) v_{\mathfrak{t}} + b_{\mathfrak{t}}(k+1) v_{s_{k+1}\mathfrak{t}} \Big)$$

We are going to compare the coefficients of $v_{\mathfrak{t}}$ in both $T_k E_{k+1} E_k v_{\mathfrak{s}} - \delta E_{k+1} E_k v_{\mathfrak{s}} + \delta E_k v_{\mathfrak{s}}$ and $T_{k+1} E_k v_{\mathfrak{s}}$. Note that $E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k) E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1) = 1$ by Lemma 4.31.

Case 1. $\mathfrak{t} = \tilde{\mathfrak{s}}$:
Since $c_{\tilde{\mathfrak{s}}}(k) c_{\tilde{\mathfrak{s}}}(k+1) = 1$, the definitions and the remarks above show that the coefficient of $v_{\mathfrak{t}}$ in $T_k E_{k+1} E_k v_{\mathfrak{s}} - \delta E_{k+1} E_k v_{\mathfrak{s}} + \delta E_k v_{\mathfrak{s}}$ is equal to

$$\begin{aligned}
&E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k) E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1) (T_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k) - \delta) + \delta E_{\mathfrak{s},\tilde{\mathfrak{s}}}(k) \\
&= E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k) \Big(\delta E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1) \frac{E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k) - 1}{c_{\tilde{\mathfrak{s}}}(k)^2 - 1} - \delta E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1) + \delta \Big) \\
&= E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k) \frac{\delta (E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1) - 1)}{c_{\tilde{\mathfrak{s}}}(k+1)^2 - 1} \\
&= E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k) T_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1)
\end{aligned}$$

which is the coefficient of $v_{\mathfrak{t}}$ in $T_{k+1} E_k v_{\mathfrak{s}}$.

Case 2. $\mathfrak{t} \overset{k}{\sim} \tilde{\mathfrak{s}}$ **and** $\mathfrak{t} \neq \tilde{\mathfrak{s}}$:
Now, $c_{\tilde{\mathfrak{s}}}(k) = c_{\mathfrak{t}}(k+2)$ and $c_{\mathfrak{t}}(k+1) = c_{\mathfrak{t}}(k)^{-1}$, so the coefficient of $v_{\mathfrak{t}}$ in $T_k E_{k+1} E_k v_{\mathfrak{s}} - \delta E_{k+1} E_k v_{\mathfrak{s}} + \delta E_k v_{\mathfrak{s}}$ is

$$\begin{aligned}
&E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k) E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1) T_{\tilde{\mathfrak{s}}\mathfrak{t}}(k) + \delta E_{\mathfrak{s}\mathfrak{t}}(k) \\
&= E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k) E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k+1) \frac{\delta E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k)}{c_{\tilde{\mathfrak{s}}}(k) c_{\mathfrak{t}}(k) - 1} + \delta E_{\mathfrak{s}\mathfrak{t}}(k) \\
&= \delta E_{\mathfrak{s}\mathfrak{t}}(k) \frac{1}{1 - c_{\tilde{\mathfrak{s}}}(k)^{-1} c_{\mathfrak{t}}(k)^{-1}} \\
&= \delta E_{\mathfrak{s}\mathfrak{t}}(k) \frac{c_{\mathfrak{t}}(k+2)}{c_{\mathfrak{t}}(k+2) - c_{\mathfrak{t}}(k+1)} \\
&= E_{\mathfrak{s}\mathfrak{t}}(k) a_{\mathfrak{t}}(k+1).
\end{aligned}$$

which is the coefficient of $v_{\mathfrak{t}}$ in $T_{k+1} E_k v_{\mathfrak{s}}$.

Case 3. $\mathfrak{t} \overset{k+1}{\sim} \tilde{\mathfrak{s}}$ **and** $\mathfrak{t} \neq \tilde{\mathfrak{s}}$:
Since $c_{\mathfrak{t}}(k) c_{\tilde{\mathfrak{s}}}(k+1) = 1$, the coefficient of $v_{\mathfrak{t}}$ in $T_k E_{k+1} E_k v_{\mathfrak{s}} - \delta E_{k+1} E_k v_{\mathfrak{s}} + \delta E_k v_{\mathfrak{s}}$

is

$$\begin{aligned}
&(a_{\mathfrak{t}}(k)-\delta)E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1)E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)\\
=&(\frac{\delta c_{\mathfrak{t}}(k+1)}{c_{\mathfrak{t}}(k+1)-c_{\mathfrak{t}}(k)}-\delta)E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1)E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)\\
=&E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)\frac{\delta E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1)}{c_{\mathfrak{t}}(k+1)c_{\tilde{\mathfrak{s}}}(k+1)-1}\\
=&E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)T_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1)
\end{aligned}$$

which is the coefficient of $v_{\mathfrak{t}}$ in $T_{k+1}E_kv_{\mathfrak{s}}$.

Now suppose that $s_k\mathfrak{t}$ is defined and let $\mathfrak{u}=s_{k+1}s_k\mathfrak{t}$ be as above. Then the coefficient of $v_{s_k\mathfrak{t}}$ in $T_kE_{k+1}E_kv_{\mathfrak{s}}-\delta E_{k+1}E_kv_{\mathfrak{s}}+\delta E_kv_{\mathfrak{s}}$ is

$$\begin{aligned}
E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1)b_{\mathfrak{t}}(k)&=\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)}\sqrt{E_{\mathfrak{t}\mathfrak{t}}(k+1)}b_{\mathfrak{t}}(k)\\
&=\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)}\sqrt{E_{\mathfrak{u}\mathfrak{u}}(k)}b_{\mathfrak{u}}(k+1)\\
&=E_{\mathfrak{s}\mathfrak{u}}(k)b_{\mathfrak{u}}(k+1),
\end{aligned}$$

where the second equality comes from (4.18)(f). As $s_k\mathfrak{t}=s_{k+1}\mathfrak{u}$ this is the coefficient of $v_{s_k\mathfrak{t}}$ in $T_{k+1}E_kv_{\mathfrak{s}}$.

In summary, we have proved that $(T_kE_{k+1}E_k-\delta E_{k+1}E_k+\delta E_k)v_{\mathfrak{s}}=T_{k+1}E_kv_{\mathfrak{s}}$. By Lemma 4.36 and Lemma 4.35(j),

$$\begin{aligned}
E_{k+1}E_kv_{\mathfrak{s}}&=(T_k^2-\delta T_k+\delta\varrho E_k)E_{k+1}E_kv_{\mathfrak{s}}\\
&=T_k(T_kE_{k+1}E_kv_{\mathfrak{s}}-\delta E_{k+1}E_kv_{\mathfrak{s}}+\delta E_kv_{\mathfrak{s}})\\
&=T_k(T_{k+1}E_kv_{\mathfrak{s}})=T_kT_{k+1}E_kv_{\mathfrak{s}},
\end{aligned}$$

and (a) follows.

In order to prove (b), we need to consider four cases as follows.

Case 1. $\mathfrak{s}_k=\mathfrak{s}_{k+2}$ and $\mathfrak{s}_{k-1}=\mathfrak{s}_{k+1}$:
We have

$$\begin{aligned}
&E_{k+1}E_kT_{k+1}v_{\mathfrak{s}}-\delta E_{k+1}E_kv_{\mathfrak{s}}+\delta E_{k+1}v_{\mathfrak{s}}\\
=&E_{k+1}E_kT_{\mathfrak{s}\mathfrak{s}}(k+1)v_{\mathfrak{s}}-\delta E_{k+1}E_kv_{\mathfrak{s}}+\delta E_{k+1}v_{\mathfrak{s}}\\
=&(T_{\mathfrak{s}\mathfrak{s}}(k+1)-\delta)E_{k+1}E_{\mathfrak{s}\mathfrak{s}}(k)v_{\mathfrak{s}}+\delta E_{k+1}v_{\mathfrak{s}}\\
=&((T_{\mathfrak{s}\mathfrak{s}}(k+1)-\delta)E_{\mathfrak{s}\mathfrak{s}}(k)+\delta)E_{k+1}v_{\mathfrak{s}}\\
=&\frac{\delta c_{\mathfrak{s}}(k+1)^2}{c_{\mathfrak{s}}(k+1)^2-1}(1-E_{\mathfrak{s}\mathfrak{s}}(k))E_{k+1}v_{\mathfrak{s}}\\
=&\frac{\delta(E_{\mathfrak{s}\mathfrak{s}}(k)-1)}{c_{\mathfrak{s}}(k)^2-1}E_{k+1}v_{\mathfrak{s}}\\
=&T_{\mathfrak{s}\mathfrak{s}}(k)E_{k+1}v_{\mathfrak{s}}=E_{k+1}T_kv_{\mathfrak{s}}.
\end{aligned}$$

Case 2. $\mathfrak{s}_k\neq\mathfrak{s}_{k+2}$ and $\mathfrak{s}_{k-1}=\mathfrak{s}_{k+1}$:
Define $\tilde{\mathfrak{s}}\in\mathscr{T}_n^{ud}(\lambda)$ to be the unique updown tableau such that $\tilde{\mathfrak{s}}\overset{k}{\sim}\mathfrak{s}$ and $\tilde{\mathfrak{s}}_k=\mathfrak{s}_{k+2}$.

Then $\tilde{\mathfrak{s}} \neq \mathfrak{s}$ and

$$\begin{aligned}
&E_{k+1}E_kT_{k+1}v_{\mathfrak{s}} - \delta E_{k+1}E_kv_{\mathfrak{s}} + \delta E_{k+1}v_{\mathfrak{s}}\\
=&E_{k+1}E_k(a_{\mathfrak{s}}(k+1)v_{\mathfrak{s}} + b_{\mathfrak{s}}(k+1)v_{s_{k+1}\mathfrak{s}}) - \delta E_{k+1}E_kv_{\mathfrak{s}}\\
=&(a_{\mathfrak{s}}(k+1) - \delta)E_{k+1}E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)v_{\tilde{\mathfrak{s}}}\\
=&\frac{\delta E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)c_{\mathfrak{s}}(k+1)}{c_{\mathfrak{s}}(k+2) - c_{\mathfrak{s}}(k+1)}E_{k+1}v_{\tilde{\mathfrak{s}}}\\
=&\frac{\delta E_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)}{c_{\mathfrak{s}}(k)c_{\tilde{\mathfrak{s}}}(k) - 1}E_{k+1}v_{\tilde{\mathfrak{s}}}\\
=&T_{\mathfrak{s}\tilde{\mathfrak{s}}}(k)E_{k+1}v_{\tilde{\mathfrak{s}}} = E_{k+1}T_kv_{\tilde{\mathfrak{s}}}
\end{aligned}$$

where the last second equality uses the facts that $c_{\mathfrak{s}}(k+1)c_{\mathfrak{s}}(k) = 1$, $c_{\mathfrak{s}}(k+2) = c_{\tilde{\mathfrak{s}}}(k)$ and $(s_{k+1}\mathfrak{s})_{k-1} \neq (s_{k+1}\mathfrak{s})_{k+1}$.

Case 3. $\mathfrak{s}_k = \mathfrak{s}_{k+2}$ **and** $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$:
Let $\tilde{\mathfrak{s}} \in \mathscr{T}_n^{ud}(\lambda)$ such that $\tilde{\mathfrak{s}} \overset{k+1}{\sim} \mathfrak{s}$ and $\tilde{\mathfrak{s}}_{k+1} = \mathfrak{s}_{k-1}$. Then

$$\begin{aligned}
&E_{k+1}E_kT_{k+1}v_{\mathfrak{s}} - \delta E_{k+1}E_kv_{\mathfrak{s}} + \delta E_{k+1}v_{\mathfrak{s}}\\
=&E_{k+1}E_kT_{\mathfrak{s}\tilde{\mathfrak{s}}}(k+1)v_{\tilde{\mathfrak{s}}} + \delta E_{k+1}v_{\mathfrak{s}}\\
=&T_{\mathfrak{s}\tilde{\mathfrak{s}}}(k+1)E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k)E_{k+1}v_{\tilde{\mathfrak{s}}} + \delta E_{k+1}v_{\mathfrak{s}}\\
=&T_{\mathfrak{s}\tilde{\mathfrak{s}}}(k+1)E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k)\sum_{\mathfrak{t}\overset{k+1}{\sim}\tilde{\mathfrak{s}}} E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1)v_{\mathfrak{t}} + \delta\sum_{\mathfrak{t}\overset{k+1}{\sim}\mathfrak{s}} E_{\mathfrak{s}\mathfrak{t}}(k+1)v_{\mathfrak{t}}
\end{aligned}$$

and $E_{k+1}T_kv_{\mathfrak{s}} = E_{k+1}(a_{\mathfrak{s}}(k)v_{\mathfrak{s}} + b_{\mathfrak{s}}(k)v_{s_k\mathfrak{s}}) = a_{\mathfrak{s}}(k)\sum_{\mathfrak{t}\overset{k+1}{\sim}\mathfrak{s}} E_{\mathfrak{s}\mathfrak{t}}(k+1)v_{\mathfrak{t}}$ since $(s_k\mathfrak{s})_k \neq (s_k\mathfrak{s})_{k+2}$ However, since $c_{\tilde{\mathfrak{s}}}(k+1) = c_{\mathfrak{s}}(k)^{-1}$, we have

$$\begin{aligned}
&T_{\mathfrak{s}\tilde{\mathfrak{s}}}(k+1)E_{\tilde{\mathfrak{s}}\tilde{\mathfrak{s}}}(k)E_{\tilde{\mathfrak{s}}\mathfrak{t}}(k+1) + \delta E_{\mathfrak{s}\mathfrak{t}}(k+1)\\
=&\frac{\delta\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k+1)}\sqrt{E_{\mathfrak{t}\mathfrak{t}}(k+1)}}{c_{\mathfrak{s}}(k+1)c_{\tilde{\mathfrak{s}}}(k+1) - 1} + \delta E_{\mathfrak{s}\mathfrak{t}}(k+1)\\
=&\frac{\delta c_{\mathfrak{s}}(k+1)}{c_{\mathfrak{s}}(k+1) - c_{\mathfrak{s}}(k)}E_{\mathfrak{s}\mathfrak{t}}(k+1) = a_{\mathfrak{s}}(k)E_{\mathfrak{s}\mathfrak{t}}(k+1).
\end{aligned}$$

So, $E_{k+1}E_kT_{k+1}v_{\mathfrak{s}} - \delta E_{k+1}E_kv_{\mathfrak{s}} + \delta E_{k+1}v_{\mathfrak{s}} = E_{k+1}T_kv_{\mathfrak{s}}$.

Case 4. $\mathfrak{s}_k \neq \mathfrak{s}_{k+2}$ **and** $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$:
Under our assumptions, we have $E_{k+1}E_kT_{k+1}v_{\mathfrak{s}} - \delta E_{k+1}E_kv_{\mathfrak{s}} + \delta E_{k+1}v_{\mathfrak{s}} = b_{\mathfrak{s}}(k+1)E_{k+1}E_kv_{s_{k+1}\mathfrak{s}}$ and $E_{k+1}T_kv_{\mathfrak{s}} = b_{\mathfrak{s}}(k)E_{k+1}v_{s_k\mathfrak{s}}$. If $(s_{k+1}\mathfrak{s})_{k-1} \neq (s_{k+1}\mathfrak{s})_{k+1}$ then $(s_k\mathfrak{s})_k \neq (s_k\mathfrak{s})_{k+2}$. So, $E_{k+1}E_kT_{k+1}v_{\mathfrak{s}} - \delta E_{k+1}E_kv_{\mathfrak{s}} + \delta E_{k+1}v_{\mathfrak{s}} = 0 = E_{k+1}T_kv_{\mathfrak{s}}$.

Suppose now that $(s_{k+1}\mathfrak{s})_{k-1} = (s_{k+1}\mathfrak{s})_{k+1}$ and let $\tilde{\mathfrak{s}} \in \mathscr{T}_n^{ud}(\lambda)$ be the unique updown tableau such that $\tilde{\mathfrak{s}} \overset{k}{\sim} s_{k+1}\mathfrak{s}$ and $\tilde{\mathfrak{s}}_k = \mathfrak{s}_{k+2}$. Set $\mathfrak{t} = s_k\mathfrak{s}$ and $\mathfrak{u} = s_{k+1}\mathfrak{s}$ and observe that the assumptions of (4.18)(f) hold, so that $b_{\mathfrak{t}}(k)\sqrt{E_{\mathfrak{t}\mathfrak{t}}(k+1)} = b_{\mathfrak{u}}(k+1)\sqrt{E_{\mathfrak{u}\mathfrak{u}}(k)}$. As $b_{\mathfrak{s}}(k) = b_{\mathfrak{t}}(k)$ and $b_{\mathfrak{s}}(k+1) = b_{\mathfrak{u}}(k+1)$. By (4.18)(d),

together with the fact that $\mathfrak{t}' \overset{k+1}{\sim} \tilde{\mathfrak{s}}$ if and only if $\mathfrak{t}' \overset{k+1}{\sim} s_k\mathfrak{s}$, we have

$$\begin{aligned}
&E_{k+1}E_kT_{k+1}v_\mathfrak{s} - \delta E_{k+1}E_kv_\mathfrak{s} + \delta E_{k+1}v_\mathfrak{s}\\
=&b_\mathfrak{s}(k+1)E_{k+1}\sum_{\mathfrak{t}'\overset{k}{\sim}\tilde{\mathfrak{s}}} E_{\mathfrak{u}\mathfrak{t}'}(k)v_{\mathfrak{t}'}\\
=&b_\mathfrak{s}(k+1)E_{\mathfrak{u}\tilde{\mathfrak{s}}}(k)E_{k+1}v_{\tilde{\mathfrak{s}}}\\
=&b_\mathfrak{s}(k+1)E_{\mathfrak{u}\tilde{\mathfrak{s}}}(k)\sum_{\mathfrak{t}'\overset{k+1}{\sim}\tilde{\mathfrak{s}}} E_{\tilde{\mathfrak{s}}\mathfrak{t}'}(k+1)v_{\mathfrak{t}'}\\
=&b_\mathfrak{s}(k)\sum_{\mathfrak{t}'\overset{k+1}{\sim}\tilde{\mathfrak{s}}} E_{\mathfrak{t}\mathfrak{t}'}(k+1)v_{\mathfrak{t}'}\\
=&b_\mathfrak{s}(k)E_{k+1}v_{s_k\mathfrak{s}} = E_{k+1}T_kv_\mathfrak{s}
\end{aligned}$$

In summary, we have proved that $E_{k+1}E_kT_{k+1}v_\mathfrak{s} - \delta E_{k+1}E_kv_\mathfrak{s} + \delta E_{k+1}v_\mathfrak{s} = E_{k+1}T_kv_\mathfrak{s}$ for any $\mathfrak{s} \in \mathscr{T}^{ud}_n(\lambda)$. So, $E_{k+1}T_k(T_{k+1}v_\mathfrak{s}) = (E_{k+1}E_kT_{k+1} - \delta E_{k+1}E_k + \delta E_{k+1})(T_{k+1}v_\mathfrak{s})$. Now, (b) follows from Lemma 4.36 and Lemma 4.35(i)(j), immediately. □

Lemma 4.38. *Suppose that $\mathfrak{s} \in \mathscr{T}^{ud}_n(\lambda)$ with $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$ and $\mathfrak{s}_k \neq \mathfrak{s}_{k+2}$, where $1 \le k < n-1$. Then $T_kT_{k+1}T_kv_\mathfrak{s} = T_{k+1}T_kT_{k+1}v_\mathfrak{s}$.*

Proof. One can verify the result without difficult if he uses the arguments in the proof of Lemma [3, 4.28]. We only give an example to illustrate it and leave the others to the reader.

Suppose that either $s_k\mathfrak{s}$ is not defined, or $s_k\mathfrak{s}$ is defined and $(s_k\mathfrak{s})_k \neq (s_k\mathfrak{s})_{k+2}$. In this case, the formulae for $T_kT_{k+1}T_kv_\mathfrak{s}$ and $T_{k+1}T_kT_{k+1}v_\mathfrak{s}$ are exactly the same as those given in the proof of [3, 4.28] up to the definitions of $a_\mathfrak{t}(k), b_t(k)$ etc. We can verify $T_kT_{k+1}T_kv_\mathfrak{s} = T_{k+1}T_kT_{k+1}v_\mathfrak{s}$ by comparing the coefficients of $v_\mathfrak{u}$ on both sides of the above equality. For example, we need to show

$$\begin{aligned}
&a_\mathfrak{s}(k)^2a_\mathfrak{s}(k+1) + a_{s_k\mathfrak{s}}(k+1)(1-a_\mathfrak{s}(k)^2+\delta a_\mathfrak{s}(k))\\
=&a_\mathfrak{s}(k)a_\mathfrak{s}(k+1)^2 + a_{s_{k+1}\mathfrak{s}}(k)(1-a_\mathfrak{s}(k+1)^2+\delta a_\mathfrak{s}(k+1))
\end{aligned}\tag{4.39}$$

when we prove that the coefficients of $v_\mathfrak{s}$ in $T_kT_{k+1}T_kv_\mathfrak{s} = T_{k+1}T_kT_{k+1}v_\mathfrak{s}$ are equal. The reader should compare the (4.39) with that given in line 4, in [3, p93]. In our case, $a_\mathfrak{s}(k) = \delta c(\beta)(c(\beta)-c(\alpha))^{-1}$, $a_\mathfrak{s}(k+1) = \delta c(\gamma)(c(\gamma)-c(\beta))^{-1}$, $a_{s_{k+1}\mathfrak{s}}(k) = a_{s_k\mathfrak{s}}(k+1) = \delta c(\gamma)(c(\gamma)-c(\alpha))^{-1}$ if we write $\mathfrak{s}_k \ominus \mathfrak{s}_{k-1} = \alpha, \mathfrak{s}_{k+1}\ominus\mathfrak{s}_k = \beta, \mathfrak{s}_{k+2}\ominus\mathfrak{s}_{k+1} = \gamma$. By direct computation, we can verify (4.39) easily. □

Lemma 4.40. *Suppose that $\mathfrak{s} \in \mathscr{T}^{ud}_n(\lambda)$ and that either $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$ and $\mathfrak{s}_k \neq \mathfrak{s}_{k+2}$, or $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$ and $\mathfrak{s}_k = \mathfrak{s}_{k+2}$, for $1 \le k < n-1$. Then $T_kT_{k+1}T_kv_\mathfrak{s} = T_{k+1}T_kT_{k+1}v_\mathfrak{s}$.*

Proof. The result can be proved by arguments given in the proof of [3, 4.29]. Since it does not involve huge computation, we include a proof here.

Case 1. $s_{k+1}\mathfrak{s}$ is defined:
Suppose first that $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$ and $\mathfrak{s}_k \neq \mathfrak{s}_{k+2}$. Then $\mathfrak{t} = s_{k+1}\mathfrak{s} \in \mathscr{T}^{ud}(\lambda)$ is well-defined. Furthermore, $\mathfrak{t}_k \neq \mathfrak{t}_{k+2}$ and $\mathfrak{t}_{k-1} \neq \mathfrak{t}_{k+1}$, so $T_kT_{k+1}T_kv_\mathfrak{t} = T_{k+1}T_kT_{k+1}v_\mathfrak{t}$

by Lemma 4.38. Now, $T_{k+1}v_{\mathfrak t}=a_{\mathfrak t}(k+1)v_{\mathfrak t}+b_{\mathfrak t}(k+1)v_{\mathfrak s}$ and $b_{\mathfrak t}(k+1)\neq 0$. Therefore

$$\begin{aligned}T_kT_{k+1}T_kv_{\mathfrak s}&=\frac{1}{b_{\mathfrak t}(k+1)}T_kT_{k+1}T_k\Big(T_{k+1}v_{\mathfrak t}-a_{\mathfrak t}(k+1)v_{\mathfrak t}\Big)\\&=\frac{1}{b_{\mathfrak t}(k+1)}\Big(T_k(T_{k+1}T_kT_{k+1})v_{\mathfrak t}-a_{\mathfrak t}(k+1)(T_kT_{k+1}T_k)v_{\mathfrak t}\Big)\\&=\frac{1}{b_{\mathfrak t}(k+1)}\Big(T_k(T_kT_{k+1}T_k)v_{\mathfrak t}-a_{\mathfrak t}(k+1)(T_{k+1}T_kT_{k+1})v_{\mathfrak t}\Big)\end{aligned}$$

by Lemma 4.38. Hence, using Lemma 4.36 twice,

$$\begin{aligned}T_kT_{k+1}T_kv_{\mathfrak s}&=\frac{1}{b_{\mathfrak t}(k+1)}\Big((1+\delta T_k-\varrho\delta E_k)T_{k+1}T_kv_{\mathfrak t}-a_{\mathfrak t}(k+1)(T_{k+1}T_kT_{k+1})v_{\mathfrak t}\Big)\\&=\frac{1}{b_{\mathfrak t}(k+1)}\Big(T_{k+1}T_k(1+\delta T_{k+1}-\varrho\delta E_{k+1})v_{\mathfrak t}-a_{\mathfrak t}(k+1)(T_{k+1}T_kT_{k+1})v_{\mathfrak t}\Big)\\&=\frac{1}{b_{\mathfrak t}(k+1)}(T_{k+1}T_kT_{k+1})\Big(T_{k+1}v_{\mathfrak t}-a_{\mathfrak t}(k+1)v_{\mathfrak t}\Big)\\&=(T_{k+1}T_kT_{k+1})v_{\mathfrak s}\end{aligned}$$

as required.

The case when $\mathfrak s_{k-1}\neq\mathfrak s_{k+1}$ and $\mathfrak s_k=\mathfrak s_{k+2}$ can be proved similarly.

Case 2. $s_{k+1}\mathfrak s$ **is not defined**:
This is equivalent to saying that the two nodes $\mathfrak s_{k+2}\ominus\mathfrak s_{k+1}$ and $\mathfrak s_{k+1}\ominus\mathfrak s_k$ are either in the same row or in the same column. Therefore, either $\mathfrak s_k\subset\mathfrak s_{k+1}\subset\mathfrak s_{k+2}$ or $\mathfrak s_k\supset\mathfrak s_{k+1}\supset\mathfrak s_{k+2}$. Note that in either case $\mathfrak s_{k-1}=\mathfrak s_{k+1}$, so we have

$$E_kv_{\mathfrak s}=\sum_{\substack{\mathfrak t\overset{k}{\sim}\mathfrak s\\ \mathfrak t\neq\mathfrak s}}E_{\mathfrak s\mathfrak t}(k)v_{\mathfrak t}+E_{\mathfrak s\mathfrak s}(k)v_{\mathfrak s}.$$

Using Lemma 4.37 and Lemma 4.35(j) twice, we have $T_kT_{k+1}T_kE_kv_{\mathfrak s}=\varrho T_kT_{k+1}E_kv_{\mathfrak s}=\varrho E_{k+1}E_kv_{\mathfrak s}=T_{k+1}E_{k+1}E_kv_{\mathfrak s}=T_{k+1}T_kT_{k+1}E_kv_{\mathfrak s}$.

Suppose that $\mathfrak t\overset{k}{\sim}\mathfrak s$ and $\mathfrak t\neq\mathfrak s$. Since the two boxes $\mathfrak s_{k+2}\ominus\mathfrak s_{k+1}$ and $\mathfrak s_{k+1}\ominus\mathfrak t_k$ belong to different rows and columns, $s_{k+1}\mathfrak t$ is well-defined and $\mathfrak t_{k-1}=\mathfrak t_{k+1}$. By Case 1, $T_{k+1}T_kT_{k+1}v_{\mathfrak t}=T_kT_{k+1}T_kv_{\mathfrak t}$. Consequently, $T_{k+1}T_kT_{k+1}E_{\mathfrak s\mathfrak s}(k)v_{\mathfrak s}=T_kT_{k+1}T_kE_{\mathfrak s\mathfrak s}(k)v_{\mathfrak s}$. Canceling the non-zero factor $E_{\mathfrak s\mathfrak s}(k)$ shows that $T_kT_{k+1}T_kv_{\mathfrak s}=T_{k+1}T_kT_{k+1}v_{\mathfrak s}$. □

Proposition 4.41. *Suppose that* $1\leq k<n-1$ *and* $\mathfrak s\in\mathscr T_n^{ud}(\lambda)$. *Then* $T_kT_{k+1}T_kv_{\mathfrak s}=T_{k+1}T_kT_{k+1}v_{\mathfrak s}$.

Proof. By Lemma 4.38 and Lemma 4.40, we need to consider the case when $\mathfrak s_{k-1}=\mathfrak s_{k+1}$ and $\mathfrak s_k=\mathfrak s_{k+2}$. By Proposition 4.37(a) and Proposition 4.35(j),

$$T_{k+1}T_kT_{k+1}E_kv_{\mathfrak s}=T_{k+1}E_{k+1}E_kv_{\mathfrak s}=\varrho E_{k+1}E_kv_{\mathfrak s}=\varrho T_kT_{k+1}E_kv_{\mathfrak s}=T_kT_{k+1}T_kE_kv_{\mathfrak s}$$

Therefore,

$$\Big(T_{k+1}T_kT_{k+1}-T_kT_{k+1}T_k\Big)\Big(E_{\mathfrak s\mathfrak s}(k)v_{\mathfrak s}+\sum_{\mathfrak t\overset{k}{\sim}\mathfrak s,\mathfrak t\neq\mathfrak s}E_{\mathfrak s\mathfrak t}(k)v_{\mathfrak t}\Big)=0.$$

Now, if $\mathfrak t\overset{k}{\sim}\mathfrak s$ and $\mathfrak t\neq\mathfrak s$ then $T_kT_{k+1}T_kv_{\mathfrak t}=T_{k+1}T_kT_{k+1}v_{\mathfrak t}$ by Lemma 4.40. Consequently, $T_kT_{k+1}T_kv_{\mathfrak s}=T_{k+1}T_kT_{k+1}v_{\mathfrak s}$ since $E_{\mathfrak s\mathfrak s}(k)\neq 0$. This completes the proof. □

Proof of Theorem 4.19. We have already checked the defining relations for $\mathscr B_{r,n}$ on $\Delta(\lambda)$. So, $\Delta(\lambda)$ is a $\mathscr B_{r,n}(\mathbf u)$–module, as we wanted to show. □

The following result shows that we can chose $u_i,q\in\mathbb R$ such that the root conditions can be satisfied in $\mathbb R$.

Lemma 4.42. *Suppose that $R=\mathbb{R}$. We choose $q, u_i \in R^+$ in such a way that $|log_{q^2}u_1| > \cdots > |log_{q^2}u_r| \geq n$ and $|log_{q^2}u_i| - |log_{q^2}u_{i+1}| \geq 2n$, and one of the following conditions holds:*

a) $q > 1$ if either $2 \nmid r$ and $\varrho^{-1} = \prod_{l=1}^{r} u_l$ or $2 \mid r$ and $\varrho^{-1} = q^{-1}\prod_{l=1}^{r} u_l$. Furthermore, we assume that $log_{q^2}u_i < 0$ if $2 \mid i$ and $log_{q^2}u_i > 0$ if $2 \nmid i$.

b) $0 < q < 1$ if either $2 \nmid r$ and $\varrho^{-1} = -\prod_{l=1}^{r} u_l$ or $2 \mid r$ and $\varrho^{-1} = -q\prod_{l=1}^{r} u_l$. Furthermore, we assume that $log_{q^2}u_i > 0$ if $2 \mid i$ and $log_{q^2}u_i < 0$ if $2 \nmid i$.

Suppose that $\mathfrak{s} \in \mathscr{T}_n^{ud}(\lambda)$ and $1 \leq k < n$. Then $1 - a_{\mathfrak{s}}(k)^2 + \delta a_{\mathfrak{s}}(k) \geq 0$, if $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$, and $E_{\mathfrak{s}\mathfrak{s}}(k) > 0$, if $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$. In particular, the Root Condition (4.18) holds if we choose positive square roots $\sqrt{b_{\mathfrak{s}}(k)} \geq 0$ and $\sqrt{E_{\mathfrak{s}\mathfrak{s}}(k)} > 0$.

Proof. We start with the case $\mathfrak{s}_{k-1} \neq \mathfrak{s}_{k+1}$. Let $\alpha = \mathfrak{s}_k \ominus \mathfrak{s}_{k-1}$ and $\beta = \mathfrak{s}_{k+1} \ominus \mathfrak{s}_k$. Define $S = \{a \in \mathbb{R}^+ | |log_{q^2}a| \geq 1\}$. By the definitions of $c(\alpha)$ and $c(\beta)$, $c(\beta)c(\alpha)^{-1} = u_i^{\pm 1}u_j^{\pm 1}q^{2(\pm k \pm l)}$ for some integers i, j, k and l. We want to prove $c(\beta)c(\alpha)^{-1} \in S$. There are two cases we need to discuss:

Case 1. $u_i^{\pm 1}u_j^{\pm 1} = 1$:
In this case, α and β are in the same component of λ. Also, both α and β are either removable nodes or addable nodes of λ. By Lemma 4.6 $c(\beta)c(\alpha)^{-1} \neq 1$. Therefore, $c(\beta)c(\alpha)^{-1} = q^{2(\pm k \pm l)} \in S$.

Case 2: $u_i^{\pm 1}u_j^{\pm 1} \neq 1$:
We have

$$\begin{aligned}|log_{q^2}(u_i^{\pm 1}u_j^{\pm 1}q^{2(\pm k \pm l)})| &= |\pm log_{q^2}u_i \pm log_{q^2}u_j \pm k \pm l| \\ &\geq |log_{q^2}u_i \pm log_{q^2}u_j| - |k \pm l| \geq 2n - |k \pm l| \geq 1.\end{aligned}$$

Hence, $c(\beta)c(\alpha)^{-1} \in S$. So,

$$\begin{aligned}1 - a_{\mathfrak{s}}(k)^2 + \delta a_{\mathfrak{s}}(k) &= \frac{(c(\beta) - q^{-2}c(\alpha))(c(\beta) - q^2c(\alpha))}{(c(\beta) - c(\alpha))^2} \\ &= \frac{(\frac{c(\beta)}{c(\alpha)} - q^{-2})(\frac{c(\beta)}{c(\alpha)} - q^2)}{(\frac{c(\beta)}{c(\alpha)} - 1)^2} > 0\end{aligned}$$

Now, we prove $E_{\mathfrak{s}\mathfrak{s}}(k) > 0$. Since we are assuming that $|log_{q^2}u_i| - |log_{q^2}u_{i+1}| \geq 2n$, $|log_{q^2}u_t \pm log_{q^2}u_{t'}| \geq 2n$ if $t' \neq t$. Therefore, the signs of $log_{q^2}u_t^{\pm 1}u_{t'}^{\pm 1}q^{2(\pm c \pm d)}$ and $\pm log_{q^2}u_t \pm log_{q^2}u_{t'}$ are the same. In other words,

$$\frac{u_t^{\pm 1}u_{t'}^{\pm 1}q^{2(\pm c \pm d)} - 1}{u_t^{\pm 1}u_{t'}^{\pm 1} - 1} > 0. \tag{4.43}$$

Similarly, we can verify

$$\frac{u_t^{\pm 2}q^{2(\pm c \pm d)} - 1}{u_t^{\pm 2} - 1} > 0. \tag{4.44}$$

Next we consider the case $\mathfrak{s}_{k-1} = \mathfrak{s}_{k+1}$. Let $\alpha = \mathfrak{s}_k \ominus \mathfrak{s}_{k-1}$ and $\lambda = \mathfrak{s}_{k-1}$. Write $\alpha = (i, j, t)$.

Let $u_tq^{2c_i}$, for $1 \leq i \leq l+1$, be the contents of the addable nodes of $\lambda^{(t)}$ and let $u_t^{-1}q^{-2d_j}$, for $1 \leq j \leq l$, be the contents of the removable nodes of $\lambda^{(t)}$. We may assume that

$$c_1 > d_1 > \cdots > c_l > d_l > c_{l+1}.$$

Let ε_t be the sign of the product of $\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)}$, where β runs over all of the addable and removable nodes of $\lambda^{(t)}$ such that $\beta \neq \alpha$. First we consider $\varepsilon_{t'}$ where $t' \neq t$.

By (4.43), $\varepsilon_{t'}$ is equal to either the sign of

$$\frac{(u_t^{-1}u_{t'}^{-1}-1)^l}{(u_t^{-1}-u_{t'}^{-1})^l}\frac{(u_t^{-1}u_{t'}-1)^{l+1}}{(u_t^{-1}-u_{t'})^{l+1}}=\frac{u_{t'}-u_t}{1-u_tu_{t'}}.$$

or the sign of

$$\frac{(u_tu_{t'}^{-1}-1)^l}{(u_t-u_{t'}^{-1})^l}\frac{(u_tu_{t'}-1)^{l+1}}{(u_t-u_{t'})^{l+1}}=\frac{1-u_tu_{t'}}{u_{t'}-u_t}.$$

It is not difficult to see that the signs of the last two equations are the same. Suppose $t'<t$ and $q>1$. There are four cases we have to discuss.

- Both t' and t are odd. Then $u_{t'}>u_t$ and $u_tu_{t'}>1$.
- t' is odd and t is even. Then $u_{t'}>u_t$ and $u_tu_{t'}>1$.
- t' is even and t is odd. Then $u_{t'}<u_t$ and $u_tu_{t'}<1$.
- Both t' and t are even. Then $u_{t'}<u_t$ and $u_tu_{t'}<1$.

So, $\varepsilon_{t'}<0$ if $t'<t$. When $t'>t$, we switch the role between t and t'. So, $\varepsilon_{t'}>0$. When $0<q<1$, we use q^{-1} instead of the previous q. Since $|log_{(q^{-1})^2}u_i|=|log_{q^2}u_i|$, we still have $\varepsilon_{t'}<0$ if $t'<t$ and $\varepsilon_{t'}>0$ if $t'>t$. Hence

$$\prod_{t'\neq t}\varepsilon_{t'}=(-1)^{t-1}.$$

Suppose $c(\alpha)=u_tq^{2c_i}$, for some i. ε_t is equal to the sign of

$$\prod_{\substack{k=1,\\k\neq i}}^{l+1}\frac{u_t^2q^{2(c_i+c_k)}-1}{u_t(q^{2c_i}-q^{2c_k})}\prod_{k=1}^{l}\frac{q^{2(c_i-d_k)}-1}{u_tq^{2c_i}-u_t^{-1}q^{-2d_k}}.$$

By (4.44), it is equal to the sign of

$$\prod_{\substack{k=1,\\k\neq i}}^{l+1}\frac{1}{q^{(2c_i-2c_k)}-1}\prod_{k=1}^{l}(q^{2(c_i-d_k)}-1)$$

so $\varepsilon_t=(-1)^{i-1}(-1)^{i-1}=1$ if $q>1$ and $\varepsilon_t=(-1)^{l-i+1}(-1)^{l-i+1}=1$ if $0<q<1$.

If $c(\alpha)=u_t^{-1}q^{-2d_j}$, for some j, then ε_t is equal to the sign of

$$\prod_{\substack{k=1,\\k\neq j}}^{l}\frac{u_t^{-2}q^{-2(d_j+d_k)}-1}{u_t^{-1}(q^{-2d_j}-q^{-2d_k})}\prod_{k=1}^{l+1}\frac{q^{2(c_k-d_j)}-1}{u_t^{-1}q^{-2d_k}-u_tq^{2c_k}}.$$

By (4.44), it is equal to the sign of

$$\prod_{\substack{k=1,\\k\neq j}}^{l}\frac{1}{q^{-2(d_j-d_k)}-1}\prod_{k=1}^{l+1}(q^{2(c_k-d_j)}-1).$$

So $\varepsilon_t=(-1)^{l-j}(-1)^{l+1-j}=-1$ if $q>1$ and $\varepsilon_t=(-1)^{j-1}(-1)^{j}=-1$ if $0<q<1$. In summary, we have proved

$$\begin{cases}\prod_{1\le t'\le r}\varepsilon_{t'}=(-1)^{t-1}, & \text{if } c(\alpha)=u_tq^{2c_i} \text{ for some } i\ ,\\ \prod_{1\le t'\le r}\varepsilon_{t'}=(-1)^{t}, & \text{if } c(\alpha)=u_t^{-1}q^{-2d_j},\ \text{for some } j\ .\end{cases}\tag{4.45}$$

We determine the sign of $E_{\mathfrak{s}\mathfrak{s}}(k)$ as follows.

Case 1. $q>1$**,** r **is odd and** $\varrho^{-1}=\prod_{l=1}^r u_i$:

$$\begin{aligned}E_{\mathfrak{s}\mathfrak{s}}(k) &= \frac{1}{\varrho c(\alpha)}(\frac{c(\alpha)-c(\alpha)^{-1}}{\delta}+1)\prod_{\beta\neq\alpha}\frac{c(\alpha)-c(\beta)^{-1}}{c(\alpha)-c(\beta)}\\ &= \frac{1}{\varrho\delta c(\alpha)^2}(c(\alpha)^2+\delta c(\alpha)-1)\frac{c(\alpha)}{\prod_{l=1}^r u_l}\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)}\\ &= \frac{1}{\delta c(\alpha)}(c(\alpha)-q^{-1})(c(\alpha)+q)\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)}\end{aligned}$$

On the other hand, under our assumption, we have

- $c(\alpha)<q^{-1}$ if either $2\mid t$ and $c(\alpha)=u_tq^{2c_i}$ or $2\nmid t$ and $c(\alpha)=u_t^{-1}q^{-2d_i}$ for some i,
- $c(\alpha)>q^{-1}$ if either $2\nmid t$ and $c(\alpha)=u_tq^{2c_i}$ or $2\mid t$ and $c(\alpha)=u_t^{-1}q^{-2d_i}$ for some i.

Since we are assuming that $q>1$, $\delta>0$. By (4.45), $E_{\mathfrak{s}\mathfrak{s}}(k)>0$ as required.

Case 2. $0<q<1$**,** r **is odd and** $\varrho^{-1}=-\prod_{l=1}^r u_i$:

$$\begin{aligned}E_{\mathfrak{s}\mathfrak{s}}(k) &= \frac{1}{\varrho c(\alpha)}(\frac{c(\alpha)-c(\alpha)^{-1}}{\delta}-1)\prod_{\beta\neq\alpha}\frac{c(\alpha)-c(\beta)^{-1}}{c(\alpha)-c(\beta)}\\ &= \frac{1}{\varrho\delta c(\alpha)^2}(c(\alpha)^2-\delta c(\alpha)-1)\frac{c(\alpha)}{\prod_{l=1}^r u_l}\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)}\\ &= \frac{-1}{\delta c(\alpha)}(c(\alpha)+q^{-1})(c(\alpha)-q)\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)}\end{aligned}$$

On the other hand, under our assumption, we have

- $c(\alpha)<q$ if either $2\mid t$ and $c(\alpha)=u_tq^{2c_i}$ or $2\nmid t$ and $c(\alpha)=u_t^{-1}q^{-2d_i}$ for some i,
- $c(\alpha)>q$ if either $2\nmid t$ and $c(\alpha)=u_tq^{2c_i}$ or $2\mid t$ and $c(\alpha)=u_t^{-1}q^{-2d_i}$ for some i.

Since we are assuming that $0<q<1$, $\delta<0$. By (4.45), $E_{\mathfrak{s}\mathfrak{s}}(k)>0$ as required.

Case 3. $q>1$**,** r **is even and** $\varrho^{-1}=q^{-1}\prod_{l=1}^r u_i$:

$$\begin{aligned}E_{\mathfrak{s}\mathfrak{s}}(k) &= \frac{1}{\varrho\delta}(1-\frac{q^2}{c(\alpha)^2})\prod_{\beta\neq\alpha}\frac{c(\alpha)-c(\beta)^{-1}}{c(\alpha)-c(\beta)}\\ &= \frac{1}{\varrho\delta c(\alpha)^2}(c(\alpha)^2-q^2)\frac{c(\alpha)}{\prod_{l=1}^r u_l}\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)}\\ &= \frac{1}{\delta q c(\alpha)}(c(\alpha)+q)(c(\alpha)-q)\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)}\end{aligned}$$

On the other hand, under our assumption, we have

- $c(\alpha)<q$ if either $2\mid t$ and $c(\alpha)=u_tq^{2c_i}$ or $2\nmid t$ and $c(\alpha)=u_t^{-1}q^{-2d_i}$ for some i,
- $c(\alpha)>q$ if either $2\nmid t$ and $c(\alpha)=u_tq^{2c_i}$ or $2\mid t$ and $c(\alpha)=u_t^{-1}q^{-2d_i}$ for some i.

Since we are assuming that $q>1$, $\delta>0$. By (4.45), $E_{\mathfrak{s}\mathfrak{s}}(k)>0$ as required.

Case 4. $0<q<1$**,** r **is even and** $\varrho^{-1}=-q\prod_{l=1}^r u_i$:

$$\begin{aligned} E_{\mathfrak{s}\mathfrak{s}}(k) &= \frac{1}{\varrho\delta}(1-\frac{1}{q^2c(\alpha)^2})\prod_{\beta\neq\alpha}\frac{c(\alpha)-c(\beta)^{-1}}{c(\alpha)-c(\beta)} \\ &= \frac{1}{\varrho\delta c(\alpha)^2}(c(\alpha)^2-q^{-2})\frac{c(\alpha)}{\prod_{l=1}^r u_l}\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)} \\ &= \frac{-q}{\delta c(\alpha)}(c(\alpha)+q^{-1})(c(\alpha)-q^{-1})\prod_{\beta\neq\alpha}\frac{c(\alpha)c(\beta)-1}{c(\alpha)-c(\beta)} \end{aligned}$$

On the other hand, under our assumption, we have

- $c(\alpha) < q^{-1}$ if either $2 \mid t$ and $c(\alpha) = u_t q^{2c_i}$ or $2 \nmid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some i,
- $c(\alpha) > q^{-1}$ if either $2 \nmid t$ and $c(\alpha) = u_t q^{2c_i}$ or $2 \mid t$ and $c(\alpha) = u_t^{-1}q^{-2d_i}$ for some i.

Since we are assuming that $0 < q < 1$, $\delta < 0$. By (4.45), $E_{\mathfrak{s}\mathfrak{s}}(k) > 0$ as required. □

5. A cellular basis of $\mathscr{B}_{r,n}(\mathbf{u})$ with odd r

Throughout this section, unless otherwise stated, we always keep the following assumption:

Assumption 5.1. *Let R be a commutative ring containing invertible elements $q, q-q^{-1}$, and u_i, $1 \le i \le r$. We also assume that $\Omega \cup \{\varrho\} \subseteq R$ is $\mathbf{u}$–admissible.*

The main purpose of this section is to construct a cellular basis for $\mathscr{B}_{r,n}$. We remark that we assume that r is odd. In other words, $r = 2p+1$ for some non-negative integer p.

In [1], Ariki and Koike have proved that Ariki-Koike algebra $\mathscr{H}_{r,n}$ is free over R. Given a non-negative integer $f \le \lfloor\frac{n}{2}\rfloor$. Then $\mathscr{H}_{r,n-2f}$ can be identified with the subalgebra of $\mathscr{H}_{r,n}$. Since we have not proved that $\mathscr{B}_{r,n}$ is free over R, we could not say $\mathscr{B}_{r,n_1}$ is a subalgebra of $\mathscr{B}_{r,n_2}$ if $n_1 < n_2$. However, there is an algebraic homomorphism from $\mathscr{B}_{r,n_1}$ to $\mathscr{B}_{r,n_2}$. So, we define $\mathscr{B}'_{r,n_1}$ to be the image of $\mathscr{B}_{r,n_1}$ in $\mathscr{B}_{r,n_2}$.

Proposition 5.2. *Given a positive integer $n \ge 2$. Let $\mathcal{E}_n = \mathscr{B}_{r,n}E_1\mathscr{B}_{r,n}$ be the two-sided ideal of $\mathscr{B}_{r,n}$ generated by E_1. Then there is a unique R–algebra isomorphism $\varepsilon_n : \mathscr{H}_{r,n} \cong \mathscr{B}_{r,n}/\mathcal{E}_n$ such that*

$$\varepsilon_n(g_i) = T_i + \mathcal{E}_n \text{ and } \varepsilon_n(y_j) = X_j + \mathcal{E}_n,$$

for $1 \le i < n$ and $1 \le j \le n$.

Proof. Let $S = \{X_j + \mathcal{E}_n, T_i + \mathcal{E}_n \mid 1 \le i \le n-1, 1 \le j \le n\}$. By Definition 2.1, S generates $\mathscr{B}_{r,n}/\mathcal{E}_n$. Therefore, ε_n is an algebraic epimorphism. We claim that $\mathscr{B}_{r,n}/\mathcal{E}_n$ is free over R with rank $r^n n!$. In fact, we consider $\mathscr{B}_{r,n}/\mathcal{E}_n$ over $R_0 := \mathbb{Z}[\mathbf{u}^{\pm1}, q^{\pm1}, \delta^{\pm1}]$ where $\delta = q-q^{-1}$. Further, we assume that $\mathbf{u}, q$ are indeterminates. We have constructed the seminormal representations for $\mathscr{B}_{r,n}$ with respect to all $\lambda \in \Lambda_r^+(n-2f)$, $0 \le f \le \lfloor\frac{n}{2}\rfloor$ under the conditions in Lemma 4.6 and (4.18). In particular, we have seminormal representations of $\mathscr{B}_{r,n}$ over $\mathbb{R}$. As $\mathbb{R}$ is not finitely generated over $\mathbb{Q}$, we can take $r+1$ algebraically independent transcedental real numbers $v_i \in \mathbb{R}$ and $\mathbf{q}$. We define $R_1 = \mathbb{Z}[v_1, v_2 \ldots, v_r, \mathbf{q}^{\pm1}, \delta^{\pm}]$ Also, we assume $\Omega \cup \{\varrho\}$ is $\mathbf{v}$-admissible. By Lemma 4.42, $\Delta(\lambda)$ are $\mathscr{B}_{r,n}/\mathcal{E}_n$-modules for all $\lambda \in \Lambda_r^+(n)$ over the field $\mathbb{R}$. By Wedderburn-Artin theorem for semisimple finite dimensional algebra,

$$\dim_{\mathbb{R}} \mathscr{B}_{r,n}/\langle E_1\rangle \ge r^n n!.$$

So, the image of an $\mathbb{R}$-basis of $\mathscr{H}_{r,n}$ has to be $\mathbb{R}$-linear independent, and hence R_1-linear independent. Therefore, $\mathscr{B}_{r,n}/\mathcal{E}_n$ is free over R_1 with rank $r^n n!$.

Note that $R_1 \cong R_0$ as rings. So, $\mathscr{B}_{r,n}(\mathfrak{u})$ over R_0 is isomorphic to $\mathscr{B}_{r,n}(\mathbf{v})$ over R_1 as R_0-modules. The corresponding isomorphism sends u_i (resp. q) to v_i (resp. $\mathbf{q}$). So, $\mathscr{B}_{r,n}/\mathcal{E}_n$ is free over R_0 with rank $r^n n!$. By base change, it is free over an arbitrary commutative ring R. So, ε_n is an isomorphism. □

Definition 5.3. Given a non-negative integer $f \leq \lfloor \frac{n}{2} \rfloor$. Let $E^f = E_{n-1}E_{n-3}\cdots E_{n-2f+1}$ and let $\mathscr{B}^f_{r,n} = \mathscr{B}_{r,n}E^f\mathscr{B}_{r,n}$. If $f = \lfloor \frac{n}{2} \rfloor \in \mathbb{Z}$, then we set $\mathscr{B}^{f+1}_{r,n} = 0$.

By Definition 5.3, there is a filtration of two-sided ideals of $\mathscr{B}_{r,n}$ as follows:

$$\mathscr{B}_{r,n} = \mathscr{B}^0_{r,n} \supset \mathscr{B}^1_{r,n} \supset \cdots \supset \mathscr{B}^{\lfloor \frac{n}{2} \rfloor}_{r,n} \supset \mathscr{B}^{\lfloor \frac{n}{2} \rfloor+1}_{r,n} = 0.$$

For $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$ let $\pi_{f,n} : \mathscr{B}^f_{r,n} \longrightarrow \mathscr{B}^f_{r,n}/\mathscr{B}^{f+1}_{r,n}$ be the corresponding projection map of $\mathscr{B}_{r,n}$–bimodules.

Since we are assuming that $r = 2p+1$ for some non-negative integer p, we set $\mathbb{N}_r = \{-p\ldots,-1,0,1,\ldots,p\}$ and define $\mathbb{N}^{f,n}_r$ to be the set of n–tuples $\kappa = (k_1,\ldots,k_n)$ such that $k_i \in \mathbb{N}_r$ and $k_i \neq 0$ only for $i = n-2j+1$, $1 \leq j \leq f$. Thus, $X^\kappa = X^{k_{n-1}}_{n-1}X^{k_{n-3}}_{n-3}\ldots X^{k_{n-2f+1}}_{n-2f+1}$.

Lemma 5.4. *Suppose that $\kappa \in \mathbb{N}^{f,n}_r$ with $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Then $E^f X^\kappa \mathcal{E}'_{n-2f} \subset B^{f+1}_{r,n}$ where $\mathcal{E}'_{n-2f}$ is the image of $\mathcal{E}_{n-2f}$ in $\mathscr{B}_{r,n}$.*

Proof. The result follows since $E^f X^\kappa$ commutes with $\mathcal{E}'_{n-2f}$. □

By Proposition 5.2 and Lemma 5.4, there is a well-defined R-module homomorphism $\sigma_f : \mathscr{H}_{r,n-2f} \longrightarrow \mathscr{B}^f_{r,n}/\mathscr{B}^{f+1}_{r,n}$, for each non-negative integer $f \leq \lfloor \frac{n}{2} \rfloor$, such that

$$\sigma_f(h) = E^f\varepsilon_{n-2f}(h)' + \mathscr{B}^{f+1}_{r,n}, \text{ for } h \in \mathscr{H}_{r,n-2f},$$

where $\varepsilon_{n-2f}(h)'$ is the image of $\varepsilon_{n-2f}(h)$ in $\mathscr{B}_{r,n}$.

We recall the following definition in [3]. Suppose that f is a non-negative integer with $f \leq \lfloor \frac{n}{2} \rfloor$. Let $\mathfrak{B}_f$ be the subgroup of $\mathfrak{S}_n$ generated by $\{s_{n-1}\} \cup \{s_{n-2i+2}s_{n-2i+1}s_{n-2i+3}s_{n-2i+2} \mid 2 \leq i \leq f\}$. Let $\tau = \big((n-2f),(2^f)\big)$ and define

$$\mathcal{D}_{f,n} = \left\{ d \in \mathfrak{S}_n \;\middle|\; \begin{array}{c} \mathfrak{t}^\tau d = (\mathfrak{t}_1,\mathfrak{t}_2) \text{ is a row standard } \tau\text{-tableau and the} \\ \text{first column of } \mathfrak{t}_2 \text{ is increasing from top to bottom} \end{array} \right\}.$$

For any positive integers i, j, write

$$s_{i,j} = \begin{cases} s_{i-1}s_{i-2}\cdots s_j, & \text{if } i > j, \\ 1, & \text{if } i = j, \\ s_is_{i+1}\cdots s_{j-1} & \text{if } i < j. \end{cases} \quad E_{i,j} = \begin{cases} E_{i-1}E_{i-2}\cdots E_j, & \text{if } i > j, \\ 1, & \text{if } i = j, \\ E_iE_{i+1}\cdots E_{j-1} & \text{if } i < j. \end{cases}$$

Lemma 5.5. *Suppose that $0 \leq f \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$\mathcal{D}_{f,n} = \left\{ s_{n-2f+1,i_f}s_{n-2f+2,j_f}\cdots s_{n-1,i_1}s_{n,j_1} \;\middle|\; \begin{array}{c} {\scriptstyle 1\leq i_f<\cdots<i_1\leq n;} \\ {\scriptstyle 1\leq i_k<j_k\leq n-2k+2; 1\leq k\leq f} \end{array} \right\}. \tag{5.6}$$

Proof. It has been proved in [3] that $\mathcal{D}_{f,n}$ is a complete set of right coset representatives for $\mathfrak{S}_{n-2f} \times \mathfrak{B}_f$ in $\mathfrak{S}_n$. So,

$$\#\mathcal{D}_{f,n} = \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_{n-2f}||\mathfrak{B}_f|} = \frac{n!}{(n-2f)!f!2^f}.$$

Let $\mathcal{D}'_{f,n}$ be the set given in the right hand of (5.6). Then $\#\mathcal{D}'_{1,n} = \binom{n}{2}$. In general, since we are assuming that $1 \leq i_f < j_f \leq n-2f+2$, $i_f \in \{1,2,\cdots,n-2f+1\}$. There are $n+2-2f-i_f$ choices for j_f. In this case, by induction assumption,

there are $\#\mathcal{D}_{f-1,n-i_f}$ choices for the sequences $i_1, j_1, \ldots, i_{f-1}, j_{f-1}$ which satisfy the inequalities in (5.6). So,

$$\#\mathcal{D}'_{f,n} = (n-2f+1)\#\mathcal{D}'_{f-1,n-1} + \cdots + 1 \cdot \#\mathcal{D}'_{f-1,2f-1}.$$

By induction assumption on $\#\mathcal{D}'_{f-1,k}$ for $2f-1 \le k \le n-1$,

$$\#\mathcal{D}'_{f,n} = \sum_{k=2f}^{n} \frac{(k-2f+1)(k-1)!}{(k-2f+1)!(f-1)!2^{f-1}}.$$

By induction on n, we have $\#\mathcal{D}'_{f,n} = \frac{n!}{(n-2f)!f!2^f} = \#\mathcal{D}_{f,n}$. Since $\mathcal{D}_{f,n} \supseteq \mathcal{D}'_{f,n}$, $\mathcal{D}_{f,n} = \mathcal{D}'_{f,n}$. $\square$

Given two standard λ–tableaux $\mathfrak{s}, \mathfrak{t}$ for some r-partition λ. It has been defined in [9] that

$$\mathfrak{m}_{\mathfrak{s}\mathfrak{t}} = g_{d(\mathfrak{s})^{-1}} \cdot \prod_{s=2}^{r} \prod_{i=1}^{a_{s-1}} (y_i - u_s) \sum_{w \in \mathfrak{S}_\lambda} g_w \cdot g_{d(\mathfrak{t})},$$

where $a_{s-1} = |\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}|$.

For each $w \in \mathfrak{S}_n$, let $T_w = T_{i_1} \cdots T_{i_k}$ if $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression of w in $\mathfrak{S}_n$. By Matsumoto's theorem (see e.g. [12, 1.2.2]), $T_w \in \mathscr{B}_{r,n}$ is independent of the reduced expression of w.

Definition 5.7. Suppose that $\lambda \in \Lambda_r^+(n-2f)$, for some non-negative integer $f \le \lfloor \frac{n}{2} \rfloor$. For each pair $(\mathfrak{s}, \mathfrak{t})$ of standard λ–tableaux define

$$M_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})^{-1}} \cdot \prod_{s=2}^{r} \prod_{i=1}^{a_{s-1}} (X_i - u_s) \sum_{w \in \mathfrak{S}_\lambda} T_w \cdot T_{d(\mathfrak{t})}.$$

Lemma 5.8. *Suppose that $\lambda \in \Lambda_r^+(n-2f)$, for some non-negative integer $f \le \lfloor \frac{n}{2} \rfloor$. For any $\mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)$,*

a) $E^f M_{\mathfrak{s}\mathfrak{t}} = M_{\mathfrak{s}\mathfrak{t}} E^f \in \mathscr{B}_{r,n}^f$.
b) If $\kappa \in \mathbb{N}_r^{f,n}$ then $M_{\mathfrak{s}\mathfrak{t}} X^\kappa = X^\kappa M_{\mathfrak{s}\mathfrak{t}}$.
c) If w is a permutation on $\{n-2f+1, \ldots, n\}$ then $M_{\mathfrak{s}\mathfrak{t}} T_w = T_w M_{\mathfrak{s}\mathfrak{t}}$.
d) $\sigma_f(\mathfrak{m}_{\mathfrak{s}\mathfrak{t}}) = \pi_{f,n}(E^f M_{\mathfrak{s}\mathfrak{t}})$.

Definition 5.9. Given a $\lambda \in \Lambda_r^+(n-2f)$ and $0 \le f \le \lfloor \frac{n}{2} \rfloor$. Define $\mathscr{B}_{r,n}^{\trianglerighteq(f,\lambda)}$ to be the two–sided ideal of $\mathscr{B}_{r,n}$ generated by $\mathscr{B}_{r,n}^{f+1}$ and the elements

$$\{\, E^f M_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\mu) \text{ and } \mu \in \Lambda_r^+(n-2f) \text{ with } \mu \trianglerighteq \lambda \,\}.$$

We also define $\mathscr{B}_{r,n}^{\triangleright(f,\lambda)} = \sum_{\mu \triangleright \lambda} \mathscr{B}_{r,n}^{\trianglerighteq(f,\mu)}$, where in the sum $\mu \in \Lambda_r^+(n-2f)$.

Theorem 5.10. *Suppose that $\mathfrak{s} \in \mathscr{T}^{std}(\lambda)$. Let $\Delta_{\mathfrak{s}}(f,\lambda)$ be the R-submodule of $\mathscr{B}_{r,n}^{\trianglerighteq(f,\lambda)} / \mathscr{B}_{r,n}^{\triangleright(f,\lambda)}$ spanned by $\{\, E^f M_{\mathfrak{s}\mathfrak{t}} X^\kappa T_d + \mathscr{B}_{r,n}^{\triangleright(f,\lambda)} \mid (\mathfrak{t}, \kappa, d) \in \delta(f,\lambda) \,\}$, where $\delta(f,\lambda) = \{\, (\mathfrak{t}, \kappa, d) \mid \mathfrak{t} \in \mathscr{T}^{std}(\lambda), \kappa \in \mathbb{N}_r^{f,n}$ and $d \in \mathcal{D}_{f,n} \,\}$. Then $\Delta_{\mathfrak{s}}(f,\lambda)$ is a right $\mathscr{B}_{r,n}$-module.*

In order to prove Theorem 5.10, we need several Lemmas as follows.

Lemma 5.11. *Let N_1 be the R-submodule generated by $\mathscr{B}'_{r,n-2} E_{n-1} X_{n-1}^{\alpha} T_d$ where $d \in \mathcal{D}_{1,n}$ and $-p \le \alpha \le p$. Then N_1 is a right $\mathscr{B}_{r,n}$-module.*

Proof. We need to verify

$$E_{n-1} X_{n-1}^{\alpha} T_d h \in N_1 \tag{5.12}$$

for any $h \in \{T_i, E_i, X_1, 1 \le i \le n-1\}$. Since $X_i = T_{i-1} X_{i-1} T_{i-1}$, $\{T_i, E_i, X_1, 1 \le i \le n-1\}$ are algebraic generators of $\mathscr{B}_{r,n}$. Since we are assuming that $d \in \mathcal{D}_{1,n}$, by Lemma 5.5, $d = s_{n-1,i} s_{n,j}$ for some integers i, j with $1 \le i < j \le n$.

First, we verify (5.12) for $h = E_k$, $1 \le k \le n-1$. By Definition 2.1(b)(c), $E_{n-1}X_{n-1}^{\alpha}T_dE_k = E_kE_{n-1}X_{n-1}^{\alpha}T_d \in N_1$ if $k \le i-2$. When $k = i$ and $j = i+1$,

$$E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,j}E_k = E_{n-1}X_{n-1}^{\alpha}E_{n,i} = \omega_{n-1}^{(\alpha)}E_{n,i},$$

where $\omega_{n-1}^{(\alpha)} \in Z(\mathscr{B}_{r,n-2})$ is given in Lemma 4.21. For the simplification of notation, we use $\omega_{n-1}^{(\alpha)}$ to express its image in $\mathscr{B}'_{r,n-2}$ in the previous equation. Since $E_{n,i} = E_{n-1}T_{n-1,i}T_{n,i+1}$, $\omega_{n-1}^{(\alpha)}E_{n,i} \in N_1$. If $j+1 \le k$, then

$$\begin{aligned}
&E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,j}E_k\\
=&E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,k-1}E_kT_{k-1,j}\\
=&E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,k+1}E_{k-1}E_kT_{k-1,j}\\
=&E_{n-1}X_{n-1}^{\alpha}T_{n-1,k-2}E_{k-1}T_{k-2,i}T_{n,k+1}E_kT_{k-1,j}\\
=&E_{k-2}E_{n-1}X_{n-1}^{\alpha}T_{n-1,k}E_{k-1}T_{k-2,i}T_{n,k+1}E_kT_{k-1,j}\\
=&E_{k-2}T_{k-2,i}T_{n-2,k-1}^{-1}E_{n-1}X_{n-1}^{\alpha}E_{n-1,k-1}T_{n,k+1}E_kT_{k-1,j}\\
=&E_{k-2}T_{k-2,i}T_{n-2,k-1}^{-1}X_{n-2}^{-\alpha}T_{n-2,k-1}E_{n,k+1}E_kE_{k-1}E_kT_{k-1,j}\\
=&E_{k-2}T_{k-2,i}T_{n-2,k-1}^{-1}X_{n-2}^{-\alpha}T_{n-2,j}E_{n,k} \in N_1
\end{aligned}$$

It is not difficult to check (5.12) if one of the conditions holds: (1) $k = i-1$, (2) $k = i, j > i+1$, (3) $i+1 \le k \le j-2$, (4) $k = j-1$ and $j > i+1$, (5) $k = j$. We leave the details to the reader. So, (5.12) holds for $h = E_k$ for all positive integers $k \le n-1$.

Now, suppose $h = T_k$. Similarly, there are eight cases we have to discuss. We only check three cases and leave the remainder cases to the reader. Note that the result for $k \le i-2$ follows immediately from Definition 2.1(b)(c).

If $k = i$ and $j > i+1$, then

$$E_{n-1}X_{n-1}^{\alpha}T_{n-1,i+1}T_i^2T_{n,j} \overset{2.1(b)}{=} E_{n-1}X_{n-1}^{\alpha}T_{n-1,i+1}(1+\delta T_i - \varrho\delta E_i)T_{n,j} \in N_1$$

since we have already proved that (5.12) holds for $h = E_i$. Similarly, when $k = j$,

$$E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,j+1}T_j^2 = E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,j+1}(1+\delta T_j - \varrho\delta E_j) \in N_1.$$

Finally, we consider the case when $h = X_1$. By Definition 2.1(b)-(c), (5.12) holds when $i > 1$. If $i = 1$, then

$$E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,j}X_1 = E_{n-1}X_{n-1}^{\alpha+1}T_{1,n-1}^{-1}T_{n,j}.$$

Since we have verified (5.12) for $h \in \{T_i, E_i \mid 1 \le i \le n-1\}$, by Definition 2.1, (5.12) still holds when $h = T_i^{-1}$ for all $1 \le i \le n-1$. So, $E_{n-1}X_{n-1}^{\alpha}T_{n-1,i}T_{n,j}X_1 \in N_1$ if $E_{n-1}X_{n-1}^{\alpha+1} \in N_1$. By assumption, it is the case when $-p \le \alpha < p$. So, we need consider the case $\alpha = p$. We have

$$\begin{aligned}
E_{n-1}X_{n-1}^{p+1} &= E_{n-1}X_{n-1}^{p}T_{n-2}X_{n-2}T_{n-2}\\
&\overset{2.3}{=} E_{n-1}(T_{n-2}X_{n-2}^{p} - \sum_{i=1}^{p}\delta X_{n-1}^{i}(E_{n-2}-1)X_{n-2}^{p-i})X_{n-2}T_{n-2}.
\end{aligned}$$

We claim that $E_{n-1}E_{n-2}X_{n-2}^{k} \in N_1$ for all positive integers $k \le p+1$. If so, then $E_{n-1}X_{n-1}^{p+1} \in N_1$ since $E_{n-1}X_{n-1}^{i} \in N_1$ for $1 \le i \le p$, $E_{n-1}T_{n-2}X_{n-2}^{p+1} = E_{n-1}E_{n-2}X_{n-2}^{p+1}T_{n-1}^{-1}$ and $E_{n-1}X_{n-1}^{i}E_{n-2} = X_{n-2}^{-i}E_{n-1}E_{n-2}$. We remark that, in this case, we need to use (5.12) for $h \in \{T_i, E_i \mid 1 \le i \le n-1\}$, which has been proved.

In order to prove our claim, we use $E_{n-2}X_{n-2}^{p+1}$ instead of $E_{n-1}X_{n-1}^{p+1}$, and repeat the previous arguments, we see that our claim holds if $E_{n,i}X_i^k \in N_1$ for all positive integers $2 \le i \le n-1$, $0 \le k \le p$ and $0 \le k \le p+1$ when $i = 1$. Since

$\prod_{i=1}^{r}(X_1 - u_i) = 0$ and $r = 2p+1$, we can use $-p \le k \le p$ instead of $0 \le k \le p+1$ when $i = 1$.

Obviously, $E_{n,n-1}X_{n-1}^k \in N_1$. In general, $E_{n,i}X_i^k = E_{n,i+1}T_iX_i^kT_{i+1}$. By Lemma 2.3,

$$E_{n,i}X_i^k = E_{n,i+1}(X_{i+1}^kT_i + \sum_{j=1}^{k}\delta X_{i+1}^j(E_i - 1)X_i^{k-j})T_{i+1}.$$

Since we have already proved that N_1 is stable under the actions of E_j, T_j, $1 \le j \le n-1$, by our induction assumption on i and k, $E_{n,i}X_i^k \in N_1$.

When $i = 1$, we have to discuss the case when $-p \le k \le 0$. In fact, $E_{n-1}X_{n-1}^k \in N_1$ for any $-p \le k \le 0$. In general,

$$\begin{aligned} E_{n,i}X_i^k &= E_{n,i+1}T_i^{-1}X_i^kT_{i+1}^{-1} \\ &\overset{2.3}{=} E_{n,i+1}(X_{i+1}^kT_i^{-1} - \sum_{j=1}^{-k}\delta X_{i+1}^{-j}(E_i - 1)X_i^{k+j})T_{i+1}^{-1} \end{aligned}$$

By our induction assumption on i and k, we have $E_{n,i}X_i^k \in N_1$. Setting $i = 1$ yields the result as required. □

The following results can be proved by arguments in the proof of Lemma 5.11.

Corollary 5.13. *Let $d \in \mathcal{D}_{1,n}$ and let α be integers with $-p \le \alpha \le p$.*
a) $E_{n-1}X_{n-1}^{\alpha}T_dE_{n-2} \in \mathscr{B}'_{r,n-2}E_{n-1}E_{n-2}$.
b) $E_{n-1}X_{n-1}^{\alpha}T_dE_{n-1} \in \mathscr{B}'_{r,n-2}E_{n-1}$.

Lemma 5.14. *For any $k \in \mathbb{Z}$, $E_{n-1}X_{n-1}^kE_{n-3} \in N_3$ where N_3 is the R-submodule of $\mathscr{B}_{r,n}$ generated by $\mathscr{B}'_{r,n-4}E_{n-1}X_{n-1}^{\ell}E_{n-3}$ and $\mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{\ell}E_{n-1}T_w$ where $w = d_1s_{n-1,n-3}s_{n,n-2}$ for some $1 \ne d_1 \in \mathcal{D}_{1,n-2}$ and $\ell \in \mathbb{Z}$ with $|\ell| \le p$.*

Proof. Note that $E_{n-1}X_{n-1}^kE_{n-3} = E_{n-3}X_{n-3}^kE_{n-1}E_{n-2}E_{n-3}$. Applying Lemma 5.11 on $E_{n-3}X_{n-3}^k$, we can write $E_{n-3}X_{n-3}^kE_{n-1}E_{n-2}E_{n-3}$ as an R-linear combination of elements in $\mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{\ell}T_{d_1}E_{n-1}E_{n-2}E_{n-3}$ where $d_1 \in \mathcal{D}_{1,n-2}$ and $\ell \in \mathbb{Z}$ with $|\ell| \le p$. Such elements are in N_3 if $d_1 \ne 1$ since $E_{n-1}T_{d_1} = T_{d_1}E_{n-1}$ and $E_{n-1}E_{n-2}E_{n-3} = E_{n-1}T_{n-2}T_{n-1}T_{n-3}T_{n-2}$. When $d_1 = 1$, $E_{n-3}X_{n-3}^{\ell}T_{d_1}E_{n-1}E_{n-2}E_{n-3} = E_{n-1}X_{n-1}^{\ell}E_{n-3} \in N_3$. □

Lemma 5.15. *Given an integer $-p \le k \le p$. $E_{n-3}\mathscr{B}'_{r,n-2}E_{n-1}E_{n-2}X_{n-2}^k \in N_{4,k}$ where $N_{4,k}$ is the R-submodule of $\mathscr{B}_{r,n}$ generated by*

$$\left\{ \mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{\ell_1}E_{n-1}X_{n-1}^{k_1}T_{d_1}T_{d_2} \,\middle|\, \begin{array}{l} d_2 \in \{1, s_{n-2}, s_{n-2}s_{n-1}\}, \\ d_1 \in \mathcal{D}_{1,n-2}, |\ell_1| \le p, |k_1| \le |k| \end{array} \right\}.$$

Proof. Applying Lemma 2.3(3)(6) (resp. Corollary 4.27) on $E_{n-1}X_{n-1}^{\ell}T_{n-1}$ (resp. $E_{n-1}X_{n-1}^{\ell}T_{n-2}E_{n-1}$), we see that both of them can be written as R-linear combinations of elements in $\mathscr{B}'_{r,n-2}E_{n-1}X_{n-1}^m$ with $|m| \le |\ell|$. Now, one can verify $N_{4,k}T_{n-1} \subseteq N_{4,k}$ by Definition 2.1 and Lemma 5.11 for $\mathscr{B}'_{r,n-2}$ without any difficult.

Given a positive integer $k \le p$ and an element $h \in \mathscr{B}'_{r,n-2}$. Then

$$\begin{aligned} E_{n-3}hE_{n-1}E_{n-2}X_{n-2}^k &= E_{n-3}hE_{n-1}T_{n-2}X_{n-2}^kT_{n-1} \\ &\overset{2.3}{=} E_{n-3}hE_{n-1}(X_{n-1}^kT_{n-2} + \sum_{i=1}^{k}\delta X_{n-1}^i(E_{n-2} - 1)X_{n-2}^{k-i})T_{n-1} \end{aligned} \tag{5.16}$$

By Lemma 5.11 for $E_{n-3}h$, $E_{n-3}hE_{n-1}X_{n-1}^kT_{n-2}T_{n-1}\in N_{4,k}$. Also, we have

$$E_{n-3}hE_{n-1}X_{n-1}^iX_{n-2}^{k-i}T_{n-1}=E_{n-3}hX_{n-2}^{k-i}E_{n-1}X_{n-1}^iT_{n-1}\in N_{4,k}.$$

One can verify the above inclusion by using $N_{4,k}T_{n-1}\subset N_{4,k}$ and Lemma 5.11 for $E_{n-3}hX_{n-2}^{k-i}$.

Finally, by induction assumption on k (when $k=0$, the result is trivial since $E_{n-1}E_{n-2}=E_{n-1}T_{n-2}T_{n-1}$), we have

$$h_1:=E_{n-3}hX_{n-2}^{-i}E_{n-1}E_{n-2}X_{n-2}^{k-i}\in N_{4,k}$$

for all positive integers $i\le k$. Since $N_{4,k}T_{n-1}\subset N_{4,k}$, $h_1T_{n-1}\in N_{4.k}$. So, $E_{n-3}hE_{n-1}E_{n-2}X_{n-2}^k\in N_{4,k}$.

The case when $-p\le k\le 0$ can be discussed similarly. The only difference is that we have to use Lemma 2.3(4) instead of Lemma 2.3(1). We leave the details to the reader. □

Lemma 5.17. *Let N_4 be the R-submodule of $\mathscr{B}_{r,n}$ generated by $\mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{\ell}E_{n-1}X_{n-1}^kT_{d_1}T_{d_2}$ where $d_2\in\{1,s_{n-2},s_{n-2}s_{n-1}\}$, $d_1\in\mathcal{D}_{1,n-2}$, $\ell,k\in\mathbb{Z}$ with $-p\le k,\ell\le p$. Then $N_4h\subset h$ for $h\in\{T_{n-1},T_{n-2},E_{n-1},E_{n-2}\}$.*

Proof. By Corollary 5.13 and Lemma 5.11, we have $N_4E_i\subset N_4$ for $i\in\{n-1,n-2\}$. Using this result together with Lemma 2.3 and Definition 2.1(b), we have $N_4T_i\subset N_4$ for $i\in\{n-1,n-2\}$. □

Lemma 5.18. *Fix an integer ℓ with $-p\le l\le p$. Let $M_\ell\subset\mathscr{B}_{r,n}$ be the R-module generated by $\{\mathscr{B}'_{r,n-4}E_{n-1}X_{n-1}^kE_{n-3}X_{n-3}^i \mid k\in\mathbb{Z},|i|\le|\ell|\}$. Let N_4 be the R-module defined in Lemma 5.17. If $E_{n-1}X_{n-1}^{\ell'}E_{n-3}X_{n-3}^mT_{n-2}T_{n-3}\in N_4$ for all integers $|\ell'|<|\ell|$ and $|m|\le p$, then $M_\ell\subseteq N_4$.*

Proof. First, we assume that $0\le i\le|\ell|$. By Lemma 5.14, $M_\ell\subseteq N_4$ if

(1) $E_{n-1}X_{n-1}^aE_{n-3}X_{n-3}^i\in N_4$ for all $a\in\mathbb{Z}$ and $|a|\le p$,
(2) $A\in N_4$ where $A=E_{n-3}X_{n-3}^aE_{n-1}T_{d_1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^i$ for all $a\in\mathbb{Z}$, $|a|\le p$ and $1\ne d_1\in\mathcal{D}_{1,n-2}$.

By the definition of N_4, (1) holds. We prove (2) by induction on i.

Suppose $i=0$. (2) holds since $d_1s_{n-2}s_{n-3}s_{n-1}s_{n-2}\in\mathcal{D}_{2,n}$. When $i>0$,

$$\begin{aligned}A=&E_{n-3}X_{n-3}^aT_{d_1}E_{n-1}T_{n-2}X_{n-2}^iT_{n-3}T_{n-1}T_{n-2}\\&+E_{n-3}X_{n-3}^aT_{d_1}E_{n-1}T_{n-2}\sum_{j=1}^{i}\delta X_{n-2}^j(E_{n-3}-1)X_{n-3}^{i-j}T_{n-1}T_{n-2}\end{aligned}\tag{5.19}$$

We consider the second term (up to a scalar in R) of (5.19) as follows.

$$\begin{aligned}&E_{n-3}X_{n-3}^aT_{d_1}E_{n-1}T_{n-2}X_{n-2}^jE_{n-3}X_{n-3}^{i-j}T_{n-1}T_{n-2}\\=&E_{n-3}X_{n-3}^aT_{d_1}X_{n-3}^{-j}E_{n-1}E_{n-2}E_{n-3}X_{n-3}^{i-j}T_{n-2}\\=&E_{n-3}X_{n-3}^aT_{d_1}X_{n-3}^{-j}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2}\\\overset{5.12}{\in}&\sum_{\substack{-p\le a'\le p\\ d_1'\in\mathcal{D}_{1,n-2}}}\mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{a'}T_{d_1'}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2}\end{aligned}$$

If $d_1'\ne1$, then $E_{n-3}X_{n-3}^{a'}T_{d_1'}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2}\in N_4$ by induction assumption on i and Lemma 5.17.

If $d_1'=1$, we still have $E_{n-3}X_{n-3}^{a'}E_{n-1}T_{n-2}T_{n-3}T_{n-1}T_{n-2}X_{n-3}^{i-j}T_{n-2}=E_{n-1}X_{n-1}^{a'}E_{n-3}X_{n-3}^{i-j}T_{n-2}\in N_4$, by Lemma 5.17.

We consider the third term on the right hand side of (5.19). We have

$$\begin{aligned}
&E_{n-3}X_{n-3}^{a}T_{d_1}E_{n-1}T_{n-2}X_{n-2}^{j}X_{n-3}^{i-j}T_{n-1}T_{n-2}\\
=&E_{n-3}X_{n-3}^{a}T_{d_1}E_{n-1}E_{n-2}X_{n-2}^{j}X_{n-3}^{i-j}T_{n-2}\\
=&E_{n-3}X_{n-3}^{a}T_{d_1}X_{n-3}^{i-j}E_{n-1}E_{n-2}X_{n-2}^{j}T_{n-2}\in N_4
\end{aligned}$$

We remark that we use Lemma 5.15 for $E_{n-3}X_{n-3}^{a}T_{d_1}X_{n-3}^{i-j}E_{n-1}E_{n-2}X_{n-2}^{j}$ and Lemma 5.17 to get the above inclusion.

We use Lemma 2.3 to express the first term on the right hand side of (5.19) as follows:

$$E_{n-3}X_{n-3}^{a}T_{d_1}E_{n-1}(X_{n-1}^{i}T_{n-2}+\sum_{j=1}^{i}\delta X_{n-1}^{j}(E_{n-2}-1)X_{n-2}^{i-j})T_{n-3}T_{n,n-2}. \tag{5.20}$$

Since we are assuming that $d_1\neq 1$, $d_1s_{n-1,n-3}s_{n,n-2}\in\mathcal{D}_{2,n}$. So, the first term on the right hand side of (5.20) is in N_4.

By Lemma 5.11 for $E_{n-3}X_{n-3}^{a}T_{d_1}X_{n-2}^{i-j}T_{n-3}$, and Lemma 5.17,

$$E_{n-3}X_{n-3}^{a}T_{d_1}X_{n-2}^{i-j}T_{n-3}E_{n-1}X_{n-1}^{j}T_{n-1}T_{n-2}\in N_4.$$

In other words, the third term on the right hand side of (5.20) is in N_4.

In order to show that the second term on the right hand side of (5.20) is in N_4, we need to show that

$$B:=E_{n-3}X_{n-3}^{a}T_{d_1}E_{n-1}X_{n-1}^{j}E_{n-2}X_{n-2}^{i-j}T_{n-3}T_{n-1}T_{n-2}\in N_4.$$

Note that

$$E_{n-3}X_{n-3}^{a}T_{d_1}E_{n-1}X_{n-1}^{j}E_{n-2}X_{n-2}^{i-j}=E_{n-3}X_{n-3}^{a}T_{d_1}X_{n-2}^{-j}E_{n-1}E_{n-2}X_{n-2}^{i-j}.$$

By Lemma 5.15, B can be written as linear combination of elements in $\mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{u}T_{d'}E_{n-1}X_{n-1}^{v}T_{d_2}T_{n-3}T_{n-1}T_{n-2}$, where $u,v\in\mathbb{Z}$ with $|u|\leq p$, $|v|\leq i-j\leq i-1$, $d'\in\mathcal{D}_{1,n-2}$, $d_2\in\{1,s_{n-2},s_{n-2}s_{n-1}\}$. In order to finish the proof, we need to show that

$$C:=E_{n-3}X_{n-3}^{u}T_{d'}E_{n-1}X_{n-1}^{v}T_{d_2}T_{n-3}T_{n-1}T_{n-2}\in N_4. \tag{5.21}$$

There are four cases we have to discuss.

(1) $d_2=1$. By Lemma 5.11 and Lemma 5.17, $C\in N_4$ since $E_{n-3}X_{n-3}^{u}T_{d'}E_{n-1}X_{n-1}^{v}T_{d_2}T_{n-3}=E_{n-3}X_{n-3}^{u}T_{d'}T_{n-3}E_{n-1}X_{n-1}^{v}$.
(2) $d_2=s_{n-2}$ and $d'\neq 1$. $C\in N_4$ since $d's_{n-2}s_{n-3}s_{n-1}s_{n-2}\in\mathcal{D}_{2,n}$.
(3) $d_2=s_{n-2}$ and $d'=1$. By Lemma 5.17 and our assumption $C\in N_4$ since $C=E_{n-3}X_{n-3}^{u}E_{n-1}X_{n-1}^{v}T_{n-2}T_{n-3}T_{n-1}T_{n-2}$ and $|v|\leq i-1\leq \ell-1$.
(4) $d_2=s_{n-2}s_{n-1}$: (5.21) follows from Lemma 5.17 and the result for $d_2=s_{n-2}$.

This completes the proof of the result under the assumption $0\leq i\leq|\ell|$. When $-|\ell|\leq i\leq 0$, One can verify the result similarly by induction on i. Note that the result holds when $i=0$. We remark that we have to use Lemma 2.3(5) instead of Lemma 2.3(1). We also need $N_4T_i^{-1}\subset N_4$ for $i=n-2,n-1$ which follows from Lemma 5.17 and Definition 2.1(b), immediately. We leave the details to the reader. □

Lemma 5.22. *Fix an integer ℓ with $-p\leq\ell\leq p$. Let N_4 be the R-module defined in Lemma 5.17. If $E_{n-1}X_{n-1}^{\ell'}E_{n-3}X_{n-3}^{m}T_{n-2}T_{n-3}\in N_4$ for all integers $|\ell'|<|\ell|$ and $|m|\leq p$, then $E_{n-3}X_{n-3}^{a}E_{n-1}E_{n-2}X_{n-2}^{b}T_{n-3}\in N_4$ with $a\in\mathbb{Z}$ and $|b|\leq|\ell|$.*

Proof. First, we assume that $0 \le b \le |\ell|$. Let $h = E_{n-3}X_{n-3}^{a}E_{n-1}E_{n-2}X_{n-2}^{b}T_{n-3}$. By Lemma 2.3(1), we have

$$h = E_{n-3}X_{n-3}^{a}E_{n-1}E_{n-2}(T_{n-3}X_{n-3}^{b} - \sum_{i=1}^{b}\delta X_{n-2}^{i}(E_{n-3}-1)X_{n-3}^{b-i}). \tag{5.23}$$

The first term on the right hand side of (5.23) is equal to

$$E_{n-3}X_{n-3}^{a}E_{n-1}E_{n-2}E_{n-3}X_{n-3}^{b}T_{n-2}^{-1} = E_{n-1}X_{n-1}^{a}E_{n-3}X_{n-3}^{b}T_{n-2}^{-1} \tag{5.24}$$

which is in N_4 by Lemmas 5.17–5.18.

The second term on the right hand side of (5.23) (up to a scalar) is equal to $E_{n-3}X_{n-3}^{a-i}E_{n-1}E_{n-2}E_{n-3}X_{n-3}^{b-i}$, which is in N_4 by (5.24). Finally, by Lemma 5.15, the last term is in N_4.

When $b < 0$, one can verify the result similarly. We remark that we have to use Lemma 2.3(4) instead of Lemma 2.3(1). □

Proposition 5.25. $E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-1}^{\ell}T_{n-2}T_{n-3} \in N_4$ *for all integers* k, ℓ *with* $-p \le k, \ell \le p$.

Proof. Let $h = E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-1}^{\ell}T_{n-2}T_{n-3}$. We prove $h \in N_4$ by induction on $|\ell|$.

If $\ell = 0$, then

$$\begin{aligned} h =& E_{n-3}X_{n-3}^{k}E_{n-1}T_{n-2}T_{n-3} = E_{n-3}X_{n-3}^{k}E_{n-1}E_{n-2}T_{n-1}^{-1}T_{n-3} \\ =& E_{n-1}X_{n-1}^{k}E_{n-3}E_{n-2}T_{n-3}T_{n-1}^{-1} = E_{n-1}X_{n-1}^{k}E_{n-3}T_{n-2}^{-1}T_{n-1}^{-1} \in N_4 \end{aligned}$$

by Lemma 5.17. If $\ell > 0$, by Lemma 2.3(1), we have

$$h = E_{n-3}X_{n-3}^{k}E_{n-1}(T_{n-2}X_{n-2}^{\ell} - \sum_{i=1}^{\ell}\delta X_{n-1}^{i}(E_{n-2}-1)X_{n-2}^{\ell-i})T_{n-3} \tag{5.26}$$

The first term on the right hand side (5.26) is equal to

$$E_{n-3}X_{n-3}^{k}E_{n-1}T_{n-2}X_{n-2}^{\ell}T_{n-3} = E_{n-3}X_{n-3}^{k}E_{n-1}E_{n-2}X_{n-2}^{\ell}T_{n-3}T_{n-1}^{-1}$$

which is in N_4 by our induction assumption on for all integers $\le \ell - 1$, together with Lemma 5.22 and Lemma 5.17.

By induction and Lemma 5.22, $E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-1}^{i}E_{n-2}X_{n-2}^{\ell-i}T_{n-3} = E_{n-3}X_{n-3}^{k+i}E_{n-1}E_{n-2}X_{n-2}^{\ell-i}T_{n-3} \in N_4$. So, The second term on the right hand side (5.26) is in N_4.

Finally, we consider the third term on the right hand side (5.26). However, this term is equal to $E_{n-3}X_{n-3}^{k}X_{n-2}^{\ell-i}T_{n-3}E_{n-1}X_{n-1}^{i}$, which is in N_4 by Lemma 5.11.

When $\ell < 0$, one can verify the result similarly. We remark that we have to use Lemma 2.3(4) instead of Lemma 2.3(1). □

Proposition 5.27. *Let* N_2 *be the* R*-submodule of* $\mathscr{B}_{r,n}$ *generated by* $\mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-1}^{\ell}T_d$ *where* $-p \le k, \ell \le p$ *and* $d \in \mathcal{D}_{2,n}$. *Then* N_2 *is a right* $\mathscr{B}_{r,n}$*-module.*

Proof. Applying Lemma 5.11 twice, $E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-1}^{\ell}T_d h$, for $h \in \mathscr{B}_{r,n}$, can be written as an R-linear combination of elements

$$\mathscr{B}'_{r,n}E_{n-3}X_{n-3}^{k_1}T_{n-3,i_4}T_{n-2,i_3}E_{n-1}X_{n-1}^{\ell_1}T_{n-1,i_2}T_{n,i_1}$$

where $i_1, i_2, i_3, i_4, k_1, \ell_1 \in \mathbb{Z}$ with $i_2 < i_1$, $i_4 < i_3$, $-p \le k_1, \ell_1 \le p$. In order to show that $E_{n-3}X_{n-3}^{k}E_{n-1}X_{n-1}^{\ell}T_d h \in N_2$, we need to show

$$A := E_{n-3}X_{n-3}^{k_1}T_{n-3,i_4}T_{n-2,i_3}E_{n-1}X_{n-1}^{\ell_1}T_{n-1,i_2}T_{n,i_1} \in N_2. \tag{5.28}$$

We are going to prove (5.28) by induction on i_2.

We assume that $i_4 \geq i_2$. Otherwise, $A \in N_2$ and (5.28) follows. In particular, (5.28) holds for $i_2 \in \{n-1, n-2\}$. By Lemma 5.11 and induction on i_2,

$$E_{n-3}\mathscr{B}'_{r,n-2}E_{n-1}X_{n-1}^{j}T_{n-1,i'_2}T_{n,i'_1} \in N_2 \tag{5.29}$$

for all integers i, j, i'_2, i'_1 with $i'_2 < i'_1$, $i, j \in \mathbb{Z}$, $-p \leq j \leq p$ and $i'_2 > i_2$.

Since we are assuming that $i_4 \geq i_2$ and $i_2 < n-2$,

$$A = E_{n-3}X_{n-3}^{k_1}E_{n-1}X_{n-1}^{\ell_1}T_{n-1,i_2}T_{n-2,i_4+1}T_{n-1,i_3+1}T_{n,i_1}.$$

By Proposition 5.25, A is in the R-submodule of $\mathscr{B}_{r,n}$ generated by

$$\mathscr{B}'_{r,n-4}E_{n-3}X_{n-3}^{k_2}T_{d_1}E_{n-1}X_{n-1}^{\ell_2}T_{d_2}T_{n-3,i_2}T_{n-2,i_4+1}T_{n-1,i_3+1}T_{n,i_1}$$

where $d_1 \in \mathcal{D}_{1,n-2}$, $d_2 \in \{s_{n-2}, s_{n-2}s_{n-1}, 1\}$ and $-p \leq k_2, \ell_2 \leq p$. In order to prove (5.28), it suffices to prove

$$B := E_{n-3}X_{n-3}^{k_2}T_{d_1}E_{n-1}X_{n-1}^{\ell_2}T_{d_2}T_{n-3,i_2}T_{n-2,i_4+1}T_{n-1,i_3+1}T_{n,i_1} \in N_2 \tag{5.30}$$

There are three cases we have to discuss.

Case 1. $d_2 = 1$:
We have $B = E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}T_{n-2,i_4+1}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_3+1}T_{n,i_1}$ which is in N_2 by (5.29) if $i_3+1 < i_1$. When $i_3+1 \geq i_1$, $T_{n-1,i_3+1}T_{n,i_1} = T_{n,i_1}T_{n,i_3+2}$ and

$$B = E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}T_{n-2,i_4+1}E_{n-1}X_{n-1}^{\ell_2}T_{n,i_1}T_{n,i_3+2}. \tag{5.31}$$

We use Lemma 2.3(3)(6) for $E_{n-1}X_{n-1}^{\ell_2}T_{n-1}$ to write B as a linear combination of elements in $E_{n-3}\mathscr{B}'_{r,n-2}E_{n-1}X_{n-1}^{j}T_{n-1,i_1}T_{n,i_3+2}$ with $-p \leq j \leq p$. By (5.29), $B \in N_2$. This completes the proof for $d_2 = 1$.

Case 2. $d_2 = s_{n-2}$:
We have $B = E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}T_{n-2,i_3}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1}T_{n,i_1}$ which is in N_2 by the result for $d_2 = 1$.

Case 3. $d_2 = s_{n-2}s_{n-1}$:
If $i_3+1 < i_1$, then

$$\begin{aligned} B &= E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1}T_{n,i_3+1}T_{n,i_1} \\ &= E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}T_{n-2,i_1-2}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1}T_{n,i_3+1} \end{aligned}$$

So, (5.30) follows from (5.29). Finally, we assume that $i_3+1 \geq i_1$. Then

$$B = E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1}T_{n-1}^2T_{n-1,i_1}T_{n,i_3+2}.$$

By Definition 2.1(b),

$$\begin{aligned} B =& E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1} \\ & \times (1 + \delta T_{n-1} - \delta\varrho E_{n-1})T_{n-1,i_1}T_{n,i_3+2}. \end{aligned} \tag{5.32}$$

The second term on the right hand side of (5.32) is equal to

$$\delta E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}T_{n-2,i_3}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1}T_{n,i_1}.$$

By our result for $d_2 = 1$, it is in N_2. Similarly, using Corollary 5.13b, Lemma 5.11, (5.29), we see that the third term on the right hand side of (5.32) is in N_2. In order to prove $B \in N_2$, it remains to prove that

$$C := E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1}T_{n-1,i_1}T_{n,i_3+2} \in M. \tag{5.33}$$

In fact, when $i_4+1 < i_1$,

$$C = E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}T_{n-2,i_1-1}E_{n-1}X_{n-1}^{\ell_2}T_{n-1,i_4+1}T_{n,i_3+2} \overset{(5.29)}{\in} N_2.$$

Suppose that $i_4+1\ge i_1$. By Definition 2.1(b),

$$\begin{aligned}C=&E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}E_{n-1}X_{n-1}^{\ell_2}\\&\times(1+\delta T_{n-2}-\delta\varrho E_{n-2})T_{n-2,i_1}T_{n-1,i_4+2}T_{n,i_3+2}.\end{aligned}\tag{5.34}$$

Note that $T_{n-1,i_1}T_{n-1,i_4+2}T_{n,i_3+2}=T_{n-2,i_4+1}T_{n-1,i_1}T_{n,i_3+2}$. By (5.29), the first and the second terms on the right of (5.34) are in N_2. The third term on the right of (5.34) (up to a scalar) is equal to

$$\begin{aligned}&E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}E_{n-1}X_{n-1}^{\ell_2}E_{n-2}T_{n-2,i_1}T_{n-1,i_4+2}T_{n,i_3+2}\\=&E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}X_{n-2}^{-\ell_2}E_{n-1}E_{n-2}T_{n-2,i_1}T_{n-1,i_4+2}T_{n,i_3+2}\\=&E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}X_{n-2}^{-\ell_2}E_{n-1}T_{n-1,i_1}T_{n,i_4+2}T_{n,i_3+2}\\=&E_{n-3}X_{n-3}^{k_2}T_{d_1}T_{n-3,i_2}X_{n-2}^{-\ell_2}T_{n-2,i_3}E_{n-1}T_{n-1,i_1}T_{n,i_4+2}\overset{5.29}{\in}N_2.\end{aligned}$$

We have proved that (5.33) holds in any case. So, $B\in N_2$ and hence (5.28) holds. □

Proof of Theorem 5.10: We claim that M is a right $\mathscr{B}_{r,n}$-module where M is the R–module generated by $\mathscr{B}'_{r,n-2f}E^fX^\kappa T_d$ with $\kappa\in\mathbb{N}_r^{f,n}$ and $d\in\mathcal{D}_{f,n}$. If so, $E^fM_{\mathfrak{s}\mathfrak{t}}X^\kappa T_dh+\mathscr{B}_{r,n}^{\rhd(f,\lambda)}$ can be written as an R-linear combination of elements $M_{\mathfrak{s}\mathfrak{t}}\mathscr{B}'_{r,n-2f}E^fX^{\kappa'}T_{d'}+\mathscr{B}_{r,n}^{\rhd(f,\lambda)}$ for $\kappa'\in\mathbb{N}_r^{f,n}$ and $d'\in\mathcal{D}_{f,n}$. Now, the result follows immediately from Lemma 5.8(d).

It remains to prove our claim. Let $d\in\mathcal{D}_{f,n}$. By Lemma 5.5,

$$d=s_{n-2f+1,i_f}s_{n-2f+2,j_f}\cdots s_{n-1,i_1}s_{n,j_1}$$

for some integers $i_1,\dots,i_f,j_1,\dots,j_f$ such that $1\le i_f<\cdots<i_1\le n$, $1\le i_k<j_k\le n-2k+2$ for $1\le k\le f$. Write $d=d_1s_{n-1,i_1}s_{n,j_1}$. For any $h\in\mathscr{B}_{r,n}$, since

$$E^fX^\kappa T_dh=\prod_{i=2}^{f}E_{n-2i+1}X_{n-2i+1}^{\kappa_{n-2i+1}}T_{d_1}E_{n-1}X_{n-1}^{k_{n-1}}T_{n-1,i_1}T_{n,j_1}h,$$

by Lemma 5.11, $E^fX^\kappa T_dh\in N$ where N is the R-submodule of $\mathscr{B}_{r,n}$ generated by

$$\prod_{i=2}^{f}E_{n-2i+1}X_{n-2i+1}^{\kappa_{n-2i+1}}T_{d_1}\mathscr{B}'_{r,n-2}E_{n-1}X_{n-1}^{\kappa'_{n-1}}T_{n-1,k_1}T_{n,\ell_1}$$

where $\kappa'_{n-1}k_1,\ell_1\in\mathbb{Z}$ with $k_1<\ell_1$ and $|\kappa'_{n-1}|\le p$. By induction assumption on $n-2$ for our claim, $E^fX^\kappa T_dh$ is in the R-submodule of $\mathscr{B}_{r,n}$ generated by

$$\mathscr{B}'_{r,n-2f}\prod_{i=1}^{f}E_{n-2i+1}X_{n-2i+1}^{\kappa'_{n-2i+1}}T_{w_1}T_{n-1,k_1}T_{n,\ell_1}$$

with $w_1=s_{n-2f+1,k_f}s_{n-2f+2,\ell_f}\cdots s_{n-3,k_2}s_{n-2,\ell_2}$ with $w_1\in\mathcal{D}_{f-1,n-2}$ and $|\kappa'_{n-2i+1}|\le p$ for $1\le i\le f$. If $w_1s_{n-1,k_1}s_{n,\ell_1}\in\mathcal{D}_{f,n}$ then our claim follows. In particular, our claim follows if $k_1\in\{n-2,n-1\}$. It remains to prove

$$A:=\prod_{i=1}^{f}E_{n-2i+1}X_{n-2i+1}^{\kappa'_{n-2i+1}}T_{w_1}T_{n-1,k_1}T_{n,\ell_1}\in M\tag{5.35}$$

for $k_2\ge k_1$. We prove it by induction on k_1. In general, we have

$$A=\prod_{i=3}^{f}E_{n-2i+1}X_{n-2i+1}^{\kappa'_{n-2i+1}}T_{d_2}B\tag{5.36}$$

where $B=E_{n-3}X_{n-3}^{\kappa'_{n-3}}E_{n-1}X_{n-1}^{\kappa'_{n-1}}T_{n-3,k_2}T_{n-2,\ell_2}T_{n-1,k_1}T_{n,\ell_1}$ and $d_2=w_1(s_{n-3,k_2}s_{n-2,\ell_2})^{-1}$.

When $k_1 \in \{n-1, n-2\}$, there is nothing to be proved since $k_2 \le n-3 < k_1$. Suppose $k_1 \le n-3$. By arguments in the proof of (5.28) for $i_4 \ge i_2$, we can write B as an R-linear combination of elements

$$C := E_{n-3}\mathscr{B}'_{r,n-2}E_{n-1}X_{n-1}^{\kappa''_{n-1}}T_{n-1,k'_1}T_{n,l'_1},$$

where $k_1 < k'_1 < l'_1$ and $-p \le \kappa''_{n-1} \le p$. By induction assumption on $\mathscr{B}_{r,n-2}$ for our claim, we can write $E^f X^\kappa T_d h$ as an R-linear combination of elements

$$\mathscr{B}'_{r,n-2f}\prod_{i=1}^{f} E_{n-2i+1}X_{n-2i+1}^{\alpha_{n-2i+1}}T_{y_1}T_{n-1,k'_1}T_{n,l'_1},$$

where $\alpha \in \mathbb{N}_r^{f,n}$, $y_1 \in \mathcal{D}_{f-1,n-2}$ and $k'_1 > k_1$. By our induction assumption on k_1, $C \in M$. This completes the proof of our claim. $\square$

Proposition 5.37. *Let $* : \mathscr{B}_{r,n} \to \mathscr{B}_{r,n}$ be the R-linear anti-involution in Lemma 2.2. Suppose $0 \le f \le \lfloor \frac{n}{2} \rfloor$ and $\lambda \in \Lambda_r^+(n-2f)$. Then $\mathscr{B}_{r,n}^{\trianglerighteq(f,\lambda)}/\mathscr{B}_{r,n}^{\triangleright(f,\lambda)}$ is spanned by the elements*

$$\{\, T_e^* X^\rho E^f M_{\mathfrak{st}} X^\kappa T_d + \mathscr{B}_{r,n}^{\triangleright(f,\lambda)} \mid (\mathfrak{t},\kappa,d),(\mathfrak{s},\rho,e) \in \delta(f,\lambda)\,\}. \tag{5.38}$$

Proof. Let W be the R–submodule of $\mathscr{B}_{r,n}^{\trianglerighteq(f,\lambda)}/\mathscr{B}_{r,n}^{\triangleright(f,\lambda)}$ spanned by the elements in (5.38). By Theorem 5.10, W is both left and right $\mathscr{B}_{r,n}$-submodule of $\mathscr{B}_{r,n}^{\trianglerighteq(f,\lambda)}/\mathscr{B}_{r,n}^{\triangleright(f,\lambda)}$. As the generators $\{E^f M_{\mathfrak{st}} + \mathscr{B}_{r,n}^{\triangleright(f,\lambda)}\}$ of $\mathscr{B}_{r,n}^{\trianglerighteq(f,\lambda)}/\mathscr{B}_{r,n}^{\triangleright(f,\lambda)}$ are contained in W, $W = \mathscr{B}_{r,n}^{\trianglerighteq(f,\lambda)}/\mathscr{B}_{r,n}^{\triangleright(f,\lambda)}$. $\square$

Definition 5.39. Let $\Lambda_{r,n}^+ = \{\,(f,\lambda) \mid 0 \le f \le \lfloor \frac{n}{2} \rfloor$ and $\lambda \in \Lambda_r^+(n-2f)\,\}$. If $(f,\lambda) \in \Lambda_{r,n}^+$ and $(\mathfrak{s},\rho,e),(\mathfrak{t},\kappa,d) \in \delta(f,\lambda)$ then we define

$$C_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)}^{(f,\lambda)} = T_e^* X^\rho E^f M_{\mathfrak{st}} X^\kappa T_d.$$

We recall the definition of cellular algebra as follows.

Definition 5.40. [16] Let R be a commutative ring and A an R–algebra. Fix a partially ordered set $\Lambda = (\Lambda, \trianglerighteq)$ and for each $\lambda \in \Lambda$ let $T(\lambda)$ be a finite set. Finally, fix $C_{\mathbf{st}}^\lambda \in A$ for all $\lambda \in \Lambda$ and $\mathbf{s},\mathbf{t} \in T(\lambda)$.

Then the triple (Λ, T, C) is a cell datum for A if:

a) $\{\, C_{\mathbf{st}}^\lambda \mid \lambda \in \Lambda$ and $\mathbf{s},\mathbf{t} \in T(\lambda)\,\}$ is an R–basis for A;

b) the R–linear map $* : A \longrightarrow A$ determined by $(C_{\mathbf{st}}^\lambda)^* = C_{\mathbf{ts}}^\lambda$, for all $\lambda \in \Lambda$ and all $\mathbf{s},\mathbf{t} \in T(\lambda)$ is an anti–isomorphism of A;

c) for all $\lambda \in \Lambda$, $\mathbf{s} \in T(\lambda)$ and $a \in A$ there exist scalars $r_{\mathbf{tu}}(a) \in R$ such that

$$C_{\mathbf{st}}^\lambda a = \sum_{\mathbf{u} \in T(\lambda)} r_{\mathbf{tu}}(a) C_{\mathbf{su}}^\lambda \pmod{A^{\triangleright\lambda}},$$

where $A^{\triangleright\lambda} = R\text{–span}\{\, C_{\mathbf{uv}}^\mu \mid \mu \triangleright \lambda$ and $\mathbf{u},\mathbf{v} \in T(\mu)\,\}$.

Furthermore, each scalar $r_{\mathbf{tu}}(a)$ is independent of $\mathbf{s}$. An algebra A is a cellular algebra if it has a cell datum and in this case we call $\{\, C_{\mathbf{st}}^\lambda \mid \mathbf{s},\mathbf{t} \in T(\lambda), \lambda \in \Lambda\,\}$ a cellular basis of A.

We recall the representation theory of cellular algebras in [16]. Every irreducible A–module arises in a unique way as the simple head of some cell module. For each $\lambda \in \Lambda$ fix $\mathbf{s} \in T(\lambda)$ and let $C_{\mathbf{t}}^\lambda = C_{\mathbf{st}}^\lambda + A^{\triangleright\lambda}$. The cell modules of A are the modules $\Delta(\lambda)$ which are the free R–modules with basis $\{\, C_{\mathbf{t}}^\lambda \mid \mathbf{t} \in T(\lambda)\,\}$. The cell module $\Delta(\lambda)$ comes equipped with a natural bilinear form ϕ_λ which is determined by the equation

$$C_{\mathbf{st}}^\lambda C_{\mathbf{t's}}^\lambda \equiv \phi_\lambda\big(C_{\mathbf{t}}^\lambda, C_{\mathbf{t'}}^\lambda\big) \cdot C_{\mathbf{ss}}^\lambda \pmod{A^{\triangleright\lambda}}.$$

The form ϕ_λ is A–invariant in the sense that $\phi_\lambda(xa, y) = \phi_\lambda(x, ya^*)$, for $x, y \in \Delta(\lambda)$ and $a \in A$. Consequently,

$$\operatorname{Rad}\Delta(\lambda) = \{\, x \in \Delta(\lambda) \mid \phi_\lambda(x, y) = 0 \text{ for all } y \in \Delta(\lambda) \,\}$$

is an A–submodule of $\Delta(\lambda)$ and $D^\lambda = \Delta(\lambda)/\operatorname{Rad}\Delta(\lambda)$ is either zero or absolutely irreducible. Graham and Lehrer have proved that $\{D^\lambda \mid D^\lambda \neq 0\}$ consists of a complete set of pairwise non-isomorphic irreducible A-modules.

Now, we use the representation theory of a cellular algebra to prove Theorem 5.41, the main result of this section.

Theorem 5.41. *Let R be a commutative ring which contains the invertible elements $q, u_1, u_2, \dots, u_r$ and $q - q^{-1}$. Suppose that $\Omega \cup \{\varrho\}$ is $\mathbf{u}$–admissible. Let $\mathscr{B}_{r,n}$ be the cyclotomic BMW algebras over R with $2 \nmid r$. Then $\mathscr{B}_{r,n}$ is free over R with*

$$\mathscr{C} = \{\, C^{(f,\lambda)}_{(\mathfrak{s},\rho,e)(\mathfrak{t},\kappa,d)} \mid (\mathfrak{s},\rho,e), (\mathfrak{t},\kappa,d) \in \delta(f,\lambda),\ \textit{where } (f,\lambda) \in \Lambda^+_{r,n} \,\}$$

as its an R-basis. Further, $\mathscr{C}$ is a cellular basis of $\mathscr{B}_{r,n}(\mathbf{u})$.

Proof. By Proposition 5.37, $\mathscr{B}_{r,n}$ is an R-module spanned by $\mathscr{C}$. First, we assume $R = R_0$ where $R_0 = \mathbb{Z}[\mathbf{u}^{\pm 1}, q^{\pm 1}, (q - q^{-1})^{-1}]$ and $\mathbf{u}, q$ are indeterminates over $\mathbb{Z}$. we prove that $\mathscr{C}$ is R_0-linear independent. As $\mathbb{R}$ is not finitely generated over $\mathbb{Q}$, we can take $r + 1$ algebraically independent transcedental real numbers $v_i \in \mathbb{R}$ and $\mathbf{q}$. We define $R_1 = \mathbb{Z}[v_1, v_2 \dots, v_r, \mathbf{q}^{\pm 1}, \delta^{\pm}]$. Then $R_1 \cong R_0$ as ring isomorphism. Therefore, $\mathscr{B}_{r,n}$ over R_0 is isomorphic to $\mathscr{B}_{r,n}$ over R_1 as R_0-algebra.

We have constructed the seminormal representations for $\mathscr{B}_{r,n}$ with respect to all $\lambda \in \Lambda^+_r(n - 2f)$, $0 \le f \le \lfloor \frac{n}{2} \rfloor$ under the conditions in Lemma 4.6 and (4.18). In particular, by Lemma 4.42, we have seminormal representations of $\mathscr{B}_{r,n}$ over $\mathbb{R}$. We remark that we are assuming that $\Omega \cup \varrho$ is $\mathbf{v}$-admissible. By arguments in the proof of Theorem 5.3 in [3], we have that $\Delta(\lambda)$ are irreducible $\mathscr{B}_{r,n}$-modules for all $\lambda \in \Lambda_r(n - 2f)$ and $0 \le f \le \lfloor \frac{n}{2} \rfloor$. Further, $\Delta(\lambda) \ncong \Delta(\mu)$ if $\lambda \neq \mu$. By Wedderburn–Artin theorem on semisimple finite dimension algebras,

$$\dim_{\mathbb{R}} \mathscr{B}_{r,n} \ge \dim_{\mathbb{R}} \mathscr{B}_{r,n}/\operatorname{Rad}\mathscr{B}_{r,n} \ge \sum_{(f,\lambda)\in\Lambda^+_{r,n}} \#\mathscr{T}^{ud}_n(\lambda)^2 = r^n(2n-1)!!,$$

the last equality follows from classical branching rule for cyclotomic Brauer algebras, which was proved in Theorem 5.11 in [22]. It was also proved in [3, 5.2]. Therefore, $\dim_{\mathbb{R}} \mathscr{B}_{r,n} = r^n(2n-1)!!$ and $\mathscr{C}$ is R_1-linear independent. So is over R_0. This shows that $\mathscr{C}$ is an R_0 basis of $\mathscr{B}_{r,n}$. By base change, $\mathscr{C}$ is an R–basis for an arbitrary commutative ring. Further, by Proposition 5.37, $\mathscr{C}$ is a cellular basis of $\mathscr{B}_{r,n}$ as required. □

In Theorem 5.41, we have assumed that r is odd. We remark that the only place we need this assumption is that we use Proposition 5.37 to prove that $\mathscr{B}_{r,n}$ is an R-module spanned by $\mathscr{C}$.

When $r = 1$, $\mathscr{B}_{r,n}$ is the usual BMW algebra defined in [8]. It has been proved in [25] that BMW algebra is cellular. Late, Enyang gave an another proof of this result in [11].

6. Classification of the irreducible $\mathscr{B}_{r,n}(\mathbf{u})$ – *modules*

In this section we assume that F is a field which contains invertible elements $u_1, \dots, u_r$, q and $q - q^{-1}$. We also assume that $\Omega \cup \{\varrho\}$ is $\mathbf{u}$-admissible. By Theorem 5.41, $\mathscr{B}_{r,n}$ is a subalgebra of $\mathscr{B}_{r,n_1}$ if $n \le n_1$. Therefore, we will identify $\mathscr{B}_{r,n}$ with $\mathscr{B}'_{r,n}$ defined in the previous section.

We are going to classify the irreducible $\mathscr{B}_{r,n}$–modules over F. We remark that we assume that r is odd.

All modules considered in this section are right modules.

Lemma 6.1. *Given a positive integer $f \le \lfloor \frac{n}{2} \rfloor$. We have $E^f \mathscr{B}_{r,n} E^f = \mathscr{B}_{r,n-2f} E^f$.*

Proof. First, we assume that $f = 1$. By Theorem 5.41, the result follows if we prove $E_{n-1} h E_{n-1} \in \mathscr{B}_{r,n-2} E_{n-1}$ for each cellular basis element $h = T_e^* X^\rho E^f M_{\mathfrak{st}} X^\kappa T_d$, where $(\mathfrak{s}, \rho, e), (\mathfrak{t}, \kappa, d) \in \delta(f, \lambda)$.

By Lemma 5.11, $E_{n-1} T_e^* X^\rho E^f M_{\mathfrak{st}} X^\kappa T_d \in N_1$ where N_1 is the R-submodule of $\mathscr{B}_{r,n}$ generated by $\mathscr{B}_{r,n-2} E_{n-1} X_{n-1}^k T_d$ where $d \in \mathcal{D}_{1,n}$ and $-p \le k \le p$. Further, by Corollary 5.13(b) for $E_{n-1} X_{n-1}^k T_d E_{n-1}$, we have

$$E_{n-1} T_e^* X^\rho E^f M_{\mathfrak{st}} X^\kappa T_d E_{n-1} \in \mathscr{B}_{r,n-2} E_{n-1}.$$

The inverse inclusion follows since $\mathscr{B}_{r,n-2} E_{n-1} = E_{n-1} \mathscr{B}_{r,n-2} E_{n-2} E_{n-1} \subset E_{n-1} \mathscr{B}_{r,n} E_{n-1}$. Using the result for $f = 1$ repeatedly, we have $E^f \mathscr{B}_{r,n} E^f = \mathscr{B}_{r,n-2f} E^f$ for all positive integers $f \le \lfloor \frac{n}{2} \rfloor$. □

It is proved in [9] that $\cup_{\lambda \in \Lambda_r^+(n)} \{\mathfrak{m}_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathscr{T}^{std}(\lambda)\}$ is a cellular basis for $\mathscr{H}_{r,n}$. Let $\Delta(\lambda)$ be the cell module of $\mathscr{H}_{r,n}$ defined by this cellular basis. Let ϕ_λ be the corresponding symmetric associative bilinear form. Let $\phi_{f,\lambda}$ be the symmetric associative bilinear form on the cell module $\Delta(f, \lambda)$ which is defined via the cellular basis of $\mathscr{B}_{r,n}$ given in Theorem 5.41.

Lemma 6.2. *Assume that $(f, \lambda) \in \Lambda_{r,n}^+$.*

a) Let $f \ne n/2$. Then $\phi_{f,\lambda} \ne 0$ if and only if $\phi_\lambda \ne 0$.

b) Let $f = n/2$, and assume that $\omega_a \ne 0$ for some non-negative integer $a \le r-1$. Then $\phi_{f,0} \ne 0$.

c) If $\omega_i = 0$ for all non-negative integers $i \le r-1$, then $\phi_{f,0} = 0$ for $f = n/2$.

Proof. (a) can be proved by arguments similar to those for [21, 3.1]. In order to prove (b), we assume that $\ell \in \mathbb{Z}$ and $k \in \mathbb{Z}^{\ge 0}$. We have

$$\begin{aligned}
&\omega_{2k+1}^{(\ell)} E_{2k+1} E_{2k-1} \cdots E_1 \\
=&E_{2k+1} X_{2k+1}^\ell E_{2k+1} E_{2k-1} \cdots E_1 \\
=&E_{2k+1} X_{2k+1}^\ell E_{2k-1} E_{2k} E_{2k-1} E_{2k+1} E_{2k-3} \cdots E_1 \\
=&E_{2k+1} E_{2k-1} X_{2k-1}^\ell E_{2k} E_{2k-1} E_{2k+1} E_{2k-3} \cdots E_1 \\
=&E_{2k-1} X_{2k-1}^\ell E_{2k+1} E_{2k} E_{2k+1} E_{2k-1} E_{2k-3} \cdots E_1 \\
=&E_{2k-1} X_{2k-1}^\ell E_{2k+1} E_{2k-1} E_{2k-3} \cdots E_1 \\
=&E_{2k+1} E_{2k-1} X_{2k-1}^\ell E_{2k-1} E_{2k-3} \cdots E_1 \\
=&E_{2k+1} E_{2k-1} E_{2k-3} \cdots E_1 X_1^\alpha E_1, \quad \text{by induction assumption} \\
=&\omega_\ell E_{2k+1} E_{2k-1} E_{2k-3} \cdots E_1.
\end{aligned}$$

Since we are assuming that $\Omega \cup \{\varrho\}$ is $\mathbf{u}$-admissible, by Theorem 5.41, $\mathscr{C}$ is an F-basis of $\mathscr{B}_{r,n}$. Since $E_{2k+1} E_{2k-1} E_{2k-3} \cdots E_1 \in \mathscr{C}$, $\omega_{2k+1}^{(\ell)} = \omega_\ell$. So,

$$\phi_{\frac{n}{2},0}(E^{\frac{n}{2}}, E^{\frac{n}{2}} X_{n-1}^\ell \cdots X_3^\ell X_1^\ell) = (\omega_\ell)^{\frac{n}{2}} \ne 0.$$

This proves (b).

Suppose that $\alpha, \beta \in \mathbb{N}_r^{f,n}$ for $f = n/2$. Using Lemma 6.1 repeatedly, we have, for any $w \in \mathfrak{S}_n$

$$E^{\frac{n}{2}} X^\alpha \cdot T_w \cdot X^\beta E^{\frac{n}{2}} = E^{\frac{n}{2}} h E_1$$

for some $h \in \mathscr{B}_{r,2}$. By direct computation, $E_1 h E_1 = 0$ for all $h \in \mathscr{B}_{r,2}$, forcing $E^{\frac{n}{2}} h E_1 = 0$. Therefore, $\phi_{\frac{n}{2},0} = 0$. This proves (c). □

Lemma 6.2 sets up a relationship between the irreducible $\mathscr{B}_{r,n}$-modules and the irreducible $\mathscr{H}_{r,n-2f}$-modules for all non-negative integers $f \le \lfloor \frac{n}{2} \rfloor$. Note that we can keep the assumption that $u_i = q^{k_i}, k_i \in \mathbb{Z}$ by using Dipper-James-Mathas's Morita equivalent theorem for $\mathscr{H}_{r,n-2f}$. In "separate condition", such a result was proved in [10]. By [2], [4] and [6], irreducible $\mathscr{H}_{r,n-2f}$-modules are indexed by $\mathbf{u}$-Kleshchev r-multipartitions of $n-2f$.

Theorem 6.3. *Suppose F is a field which contains non-zero elements $q, u_1, \ldots, u_r$ and $q - q^{-1}$. Assume that $\Omega \cup \{\varrho\}$ is $\mathbf{u}$-admissible. Let $\mathscr{B}_{r,n}$, $2 \nmid r$ be the cyclotomic BMW algebra over F.*

a) If n is odd, then the set of all pair-wise non-isomorphic irreducible $\mathscr{B}_{r,n}$-modules are indexed by (f, λ) where $0 \le f \le \lfloor \frac{n}{2} \rfloor$ and λ are $\mathbf{u}$-Kleshchev multipartitions of $n-2f$.

b) Suppose that n is an even number.

(i) If $\omega_i \ne 0$ for some non-negative integers $i \le r-1$, then the set of all pair-wise non-isomorphic irreducible $\mathscr{B}_{r,n}$-modules are indexed by (f, λ) where $0 \le f \le \frac{n}{2}$ and λ are $\mathbf{u}$-Kleshchev multipartitions of $n-2f$.

(ii) If $\omega_i = 0$ for all non-negative integers $i \le r-1$, then the set of all pair-wise non-isomorphic irreducible $\mathscr{B}_{r,n}$-modules are indexed by (f, λ) where $0 \le f < \frac{n}{2}$ and λ are $\mathbf{u}$-Kleshchev multipartitions of $n-2f$.

References

[1] S. Ariki and K. Koike, "A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations", *Adv. Math.* **106** (1994), 216–243.

[2] S. Ariki and A. Mathas, "The number of simple modules of the Hecke algebras of type $G(r,1,n)$", *Math. Z.* **233** (2000), 601–623.

[3] S. Ariki and A. Mathas, H. Rui, "Cyclotomic Nazarov-Wenzl algebras", *Nagoya Math. J.* **182** (2006), 47–134.

[4] S. Ariki, "On the decomposition numbers of the Hecke algebra of $G(m,1,n)$", *J. Math. Kyoto Univ.* **36** (1996), 789–808.

[5] ———, "On the classification of simple modules for cyclotomic Hecke algebras of type $G(m,1,n)$ and Kleshchev multipartitions", *Osaka J. Math.* **38** (2001), 827–837.

[6] ———, "Proof of the modular branching rule for cyclotomic Hecke algebras", *J. Algebra***306** (2006) 290C300.

[7] R. Brauer, "On algebras which are connected with the semisimple continuous groups", *Ann. of Math.* **38** (1937), 857–872.

[8] J. S. Birman and H. Wenzl, "Braids, link polynomials and a new algebra", *Trans. Amer. Math. Soc.* **313** (1989), 249–273.

[9] R. Dipper, G. James and A. Mathas, "Cyclotomic q–Schur algebras", *Math. Z.* **229** (1999), 385–416.

[10] J. Du and H. Rui, "Ariki-Koike algebras with semisimple bottoms", *Math. Z.* **234**, (2000), 807-830.

[11] J. Enyang, "Cellular bases for the Brauer and BMW algebras", *J. Algebra* **281** (2004), 413–449.

[12] M. Geck and G. Pfeiffer, *Characters of Finite Coxeter groups and Iwahori–Hecke Algebras*, Clarendon Press. Oxford, 2000.

[13] F. M. Goodman and H. M. Hauschild, "Affine Birman-Wenzl-Murakami Algebras and Tangles in the Solid Torus", arXiv:math.QA/0411155.

[14] ———, "Cyclotomic Birman-Wenzl-Murakami Algebras, I: freeness and realization as tangle algebras ", arXiv:math.QA/0612064.

[15] ———,"Cyclotomic Birman–Wenzl–Murakami algebras, II: Admissibility Relations and Representation theory" arXiv:math.QA/0612065.

[16] J. J. Graham and G. I. Lehrer, "Cellular algebras", *Invent. Math.* **123** (1996), 1–34.

[17] R. Häring-Oldenburg, "Cyclotomic Birman-Murakami-Wenzl algebras", *J. Pure Appl. Algebra* **161** (2001), 113–144.

[18] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second éd., Oxford Mathematical Monographs, Clarendon Press, Oxford, 1995.

[19] M. Nazarov, "Young's orthogonal form for Brauer's centralizer algebra", *J. Algebra* **182** (1996), 664–693.

[20] R. Orellana and A. Ram, “Affine braids, Markov traces and the category O”, *Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces* Mumbai (2004), V.B. Mehta ed., Tata Institute of Fundamental Research, Narosa Publishing House, Amer. Math. Soc. (2007) 423-473.

[21] H. Rui and M. Si, “On the structure of cyclotomic Nazarov–Wenzl algebras”, *J. Pure Appl. Algebra*, to appear.

[22] H. Rui and W. Yu, “On the semisimplicity of cyclotomic Brauer algebras”, *J. Algebra* **277** (2004), 187–221.

[23] Shona Yu , “The cyclotomic Birman-Murakami-Wenzl algebras”, Ph.D thesis, Sydney University, 2007.

[24] S. Wilcox and S. Yu, “The cyclotomic BMW algebra associated with two string type B braid group, ”, preprint, 2006.

[25] C. Xi, “On the quasi-heredity of Birman–Wenzl algebras”, *Adv. Math.* **154** (2000), 280–298.

[26] J. Xu, “Cyclotomic Birman-Murakami-Wenzl algebras $\mathscr{B}_{r,2}$,” thesis for Master degree, East China Normal University, June, 2006. In chinese.

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